Solving Cubic Equations: An Introduction to the Birch and Swinnerton-Dyer Conjecture

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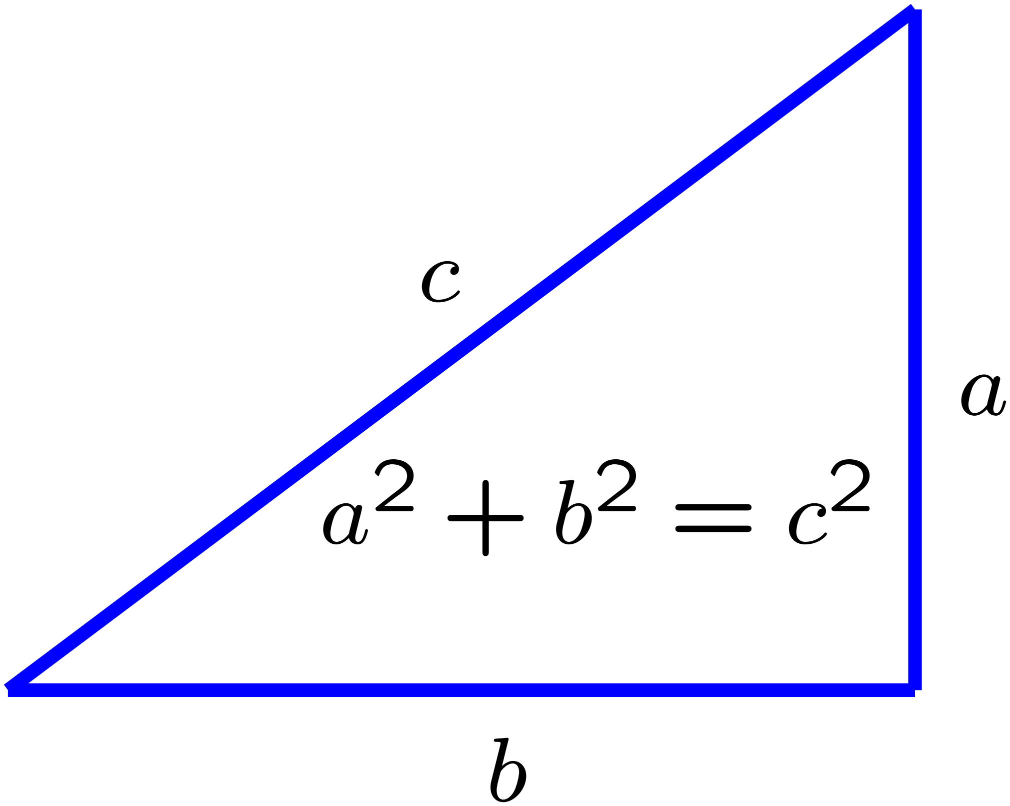
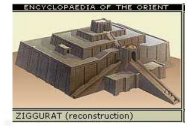
The Pythagorean Theorem



Pythagoras
Approx 569–475BC

Pythagorean Triples

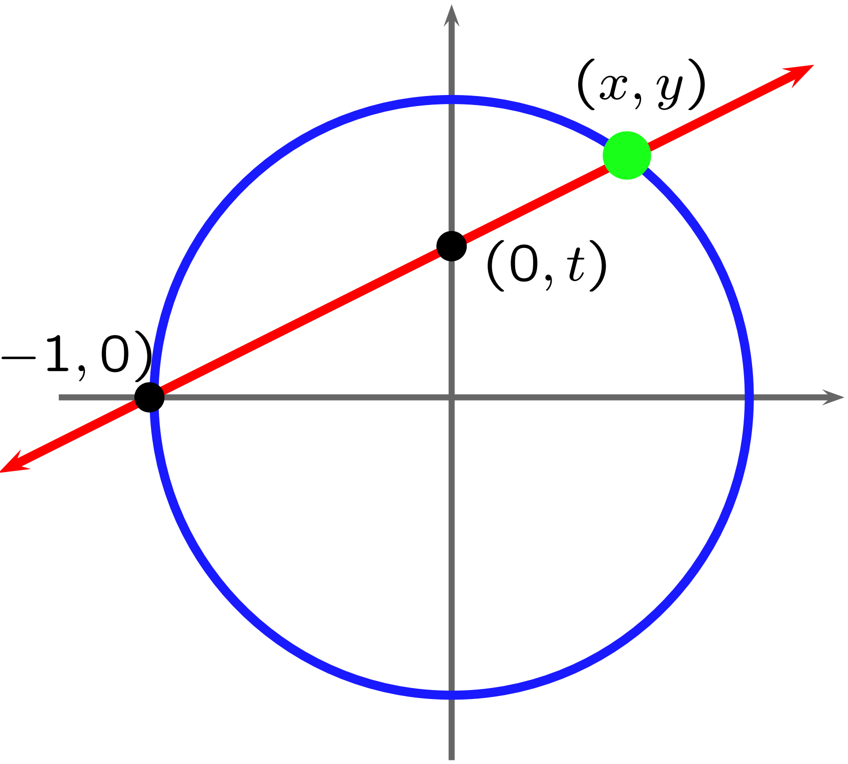


$(3, 4, 5)$
$(5, 12, 13)$
$(7, 24, 25)$
$(9, 40, 41)$
$(11, 60, 61)$
$(13, 84, 85)$
$(15, 8, 17)$
$(21, 20, 29)$
$(33, 56, 65)$
$(35, 12, 37)$
$(39, 80, 89)$
$(45, 28, 53)$
$(55, 48, 73)$
$(63, 16, 65)$
$(65, 72, 97)$
$(77, 36, 85)$
$\vdots$

Triples of integers a, b, c such that

$$a^2 + b^2 = c^2$$

Enumerating Pythagorean Triples



$$\text{Slope} = t = \frac{y}{x+1}$$

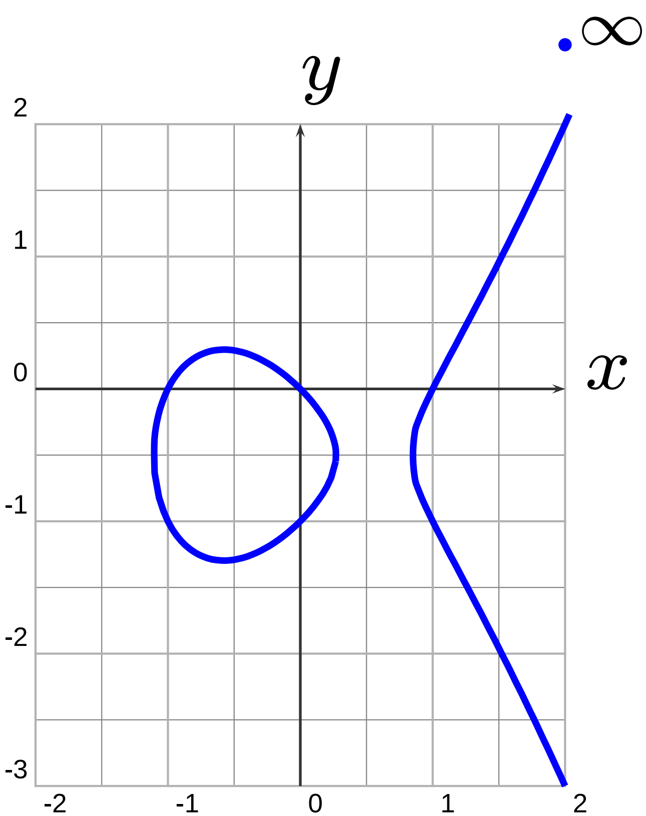
$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

If $t = \frac{r}{s}$, then $a = s^2 - r^2, \quad b = 2rs, \quad c = s^2 + r^2$
is a Pythagorean triple, and all primitive unordered triples arise in this way. **We can solve two-variable quadratic equations.**

What About Two-variable Cubic Equations?

Elliptic curve: a (smooth) plane **cubic curve** with a rational point (possibly "at infinity").



$y^2 + y = x^3 - x$

EXAMPLES

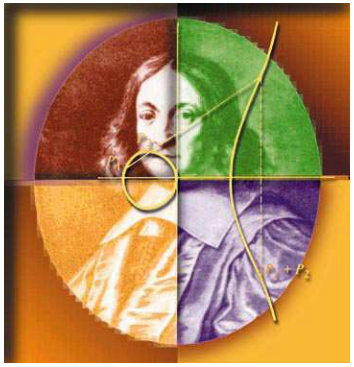
$$y^2 + y = x^3 - x$$

$$x^3 + y^3 = z^3 \text{ (homogeneous)}$$

$$y^2 = x^3 + ax + b$$

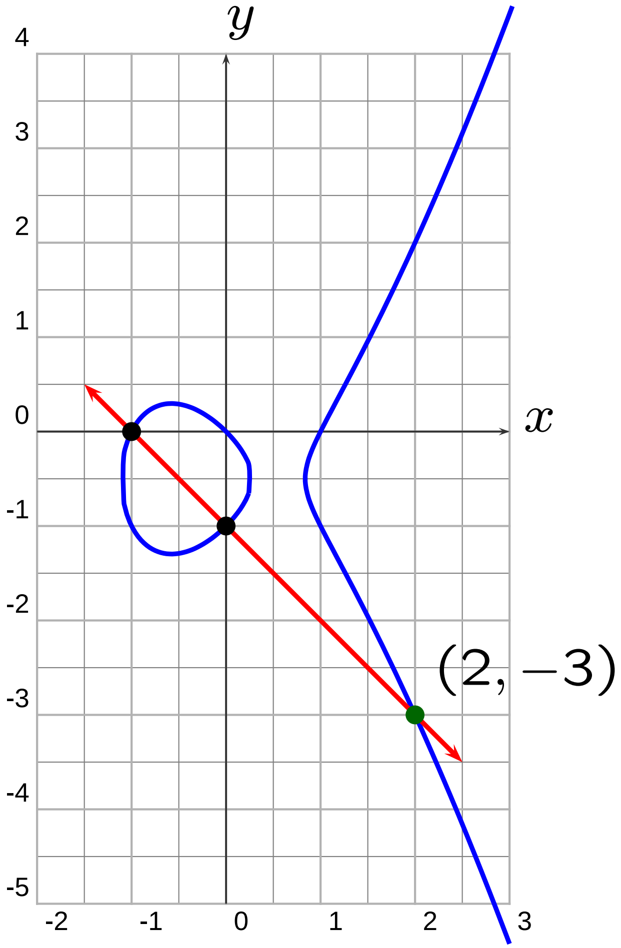
$$\cancel{3x^3 + 4y^3 + 5z^3 = 0}$$

The Secant Process



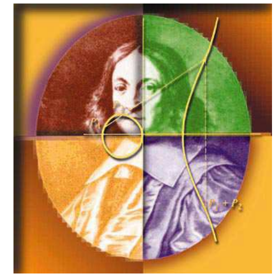
Fermat?

Obtain a third (rational!) solution from two (rational) solutions.

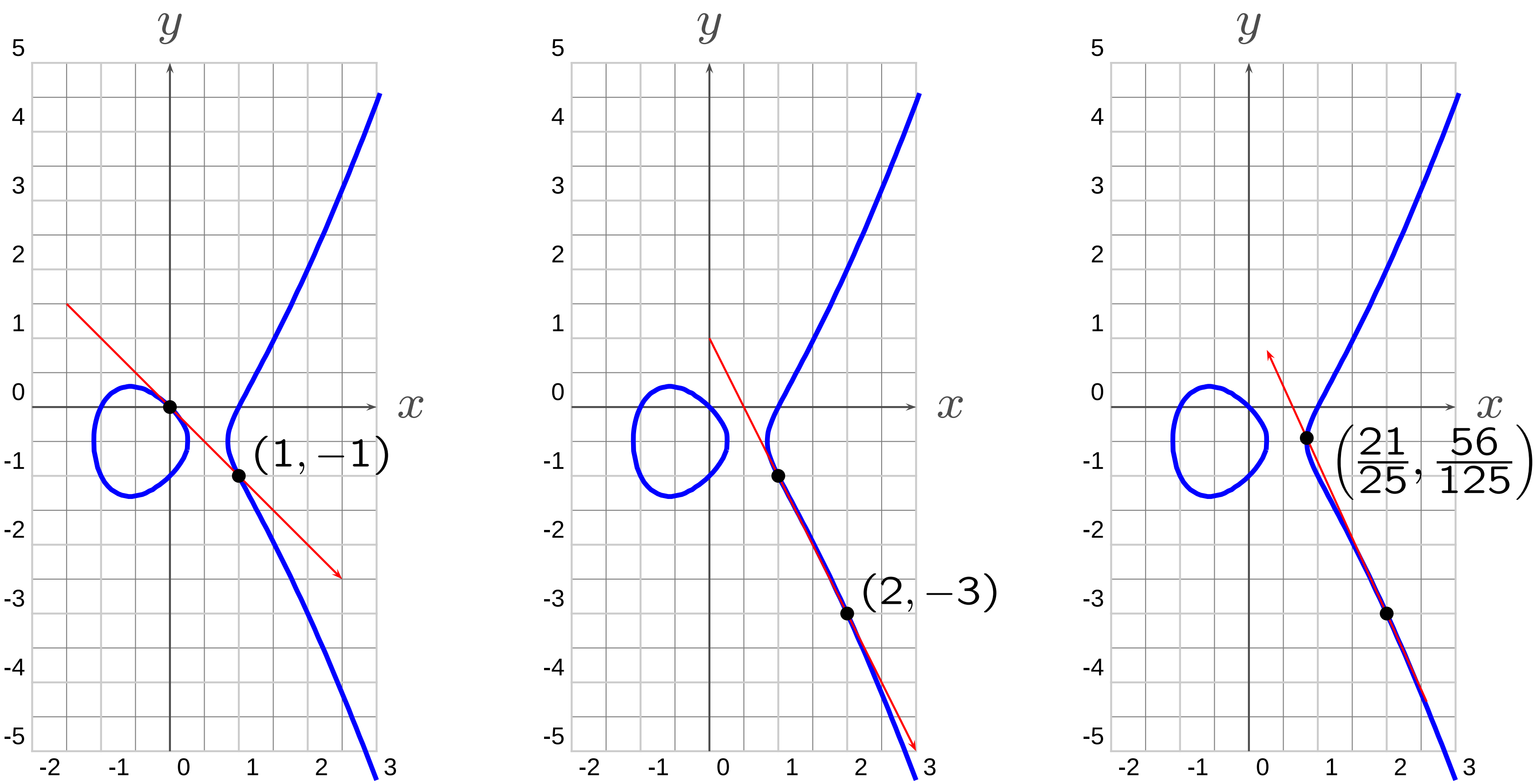


$$y^2 + y = x^3 - x$$

The Tangent Process



New rational point from a single rational point.



Iterate the Tangent Process


Fermat

$$(0, 0)$$

$$(1, -1)$$

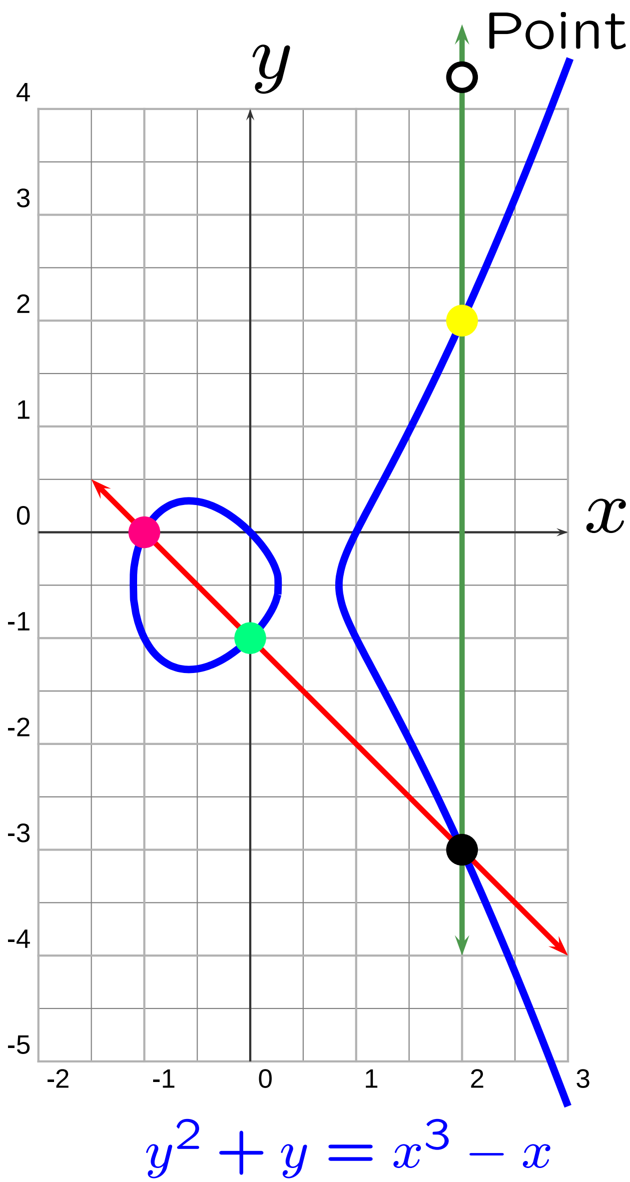
$$(2, -3)$$

$$\left(\frac{21}{25}, -\frac{56}{125}\right)$$

$$\left(\frac{480106}{4225}, \frac{332513754}{274625}\right)$$

$$\left(\frac{5313922364481462429 0821}{18700987715366274360 25}, -\frac{12282540069555885821741113162699381}{8087174560555986485289398 0186125}\right)$$

The Group Operation



$\color{#ff0077}{\bullet} \oplus \color{#00ff77}{\bullet} = \color{yellow}{\bullet}$

$$(-1, 0) \oplus (0, -1) = (2, 2)$$

The set of rational points on E forms an abelian group.

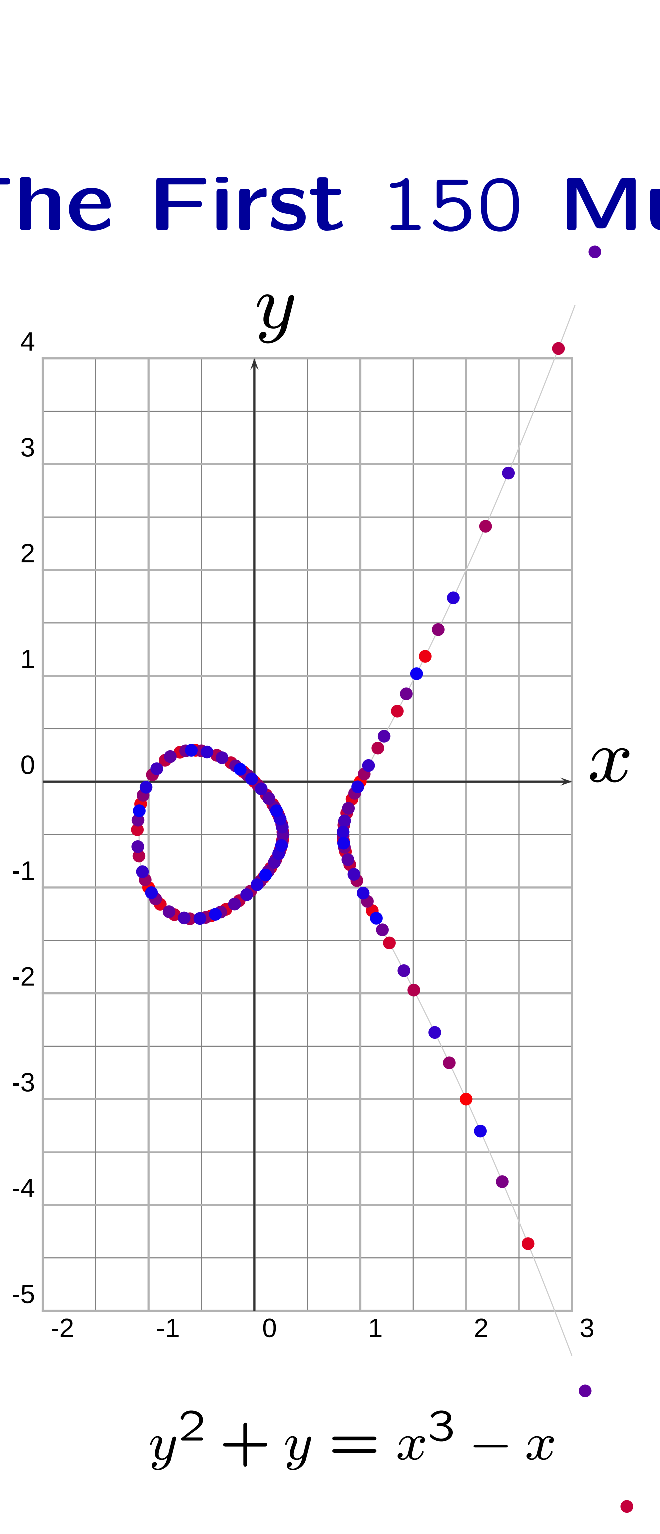
SAGE: Software for Algebra and Geometry Experimentation

```
--------------------------------------------------------------------------
  SAGE Version 0.7.8, Export Date: 2005-10-05-1650
  Distributed under the terms of the GNU General Public License (GPL)
  IPython shell -- for help type <object>?, <object>??, %magic, or help
--------------------------------------------------------------------------
sage: E = EllipticCurve([0,0,1,-1,0])
sage: E
      Elliptic Curve defined by y^2 + y = x^3 - x over Rational Field
sage: P = E([0,0])
sage: 2*P
      (1, 0)
sage: 10*P
      (161/16, -2065/64)
sage: 20*P
      (683916417/264517696, -18784454671297/4302115807744)
sage: 50*P
      (24854671723753819921380822649312751965653209957505606561/
               294187845458838221882435701984162874370013352033409888816,
      -65343698144990446428357439135977881124804221113554492507243553294512904673973173265/
       159564798621271700005828929931002008441744804573070282618997694000714045237979692864)
```

If you are interested in improving this software, contact me. I have grant funds to hire undergraduates.

http://modular.ucsd.edu/sage

The First 150 Multiples of $(0,0)$



$$y^2 + y = x^3 - x$$

(The bluer the point, the bigger the multiple.)

Fact: The group $E(\mathbb{Q})$ is generated by $(0,0)$.

In contrast, $y^2 + y = x^3 - x^2$ has only 5 rational solutions!

What is going on here?

Mordell's Theorem

Theorem (Mordell). The group $E(\mathbb{Q})$ of rational points on an elliptic curve is a **finitely generated abelian group**:

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus T,$$

with T finite.

Mazur classified the possibilities for T. It is conjectured that r can be arbitrary, but the biggest r ever found is (probably) 24.

The Simplest Solution Can Be Huge

Simplest solution to $y^2 = x^3 + 7823$:

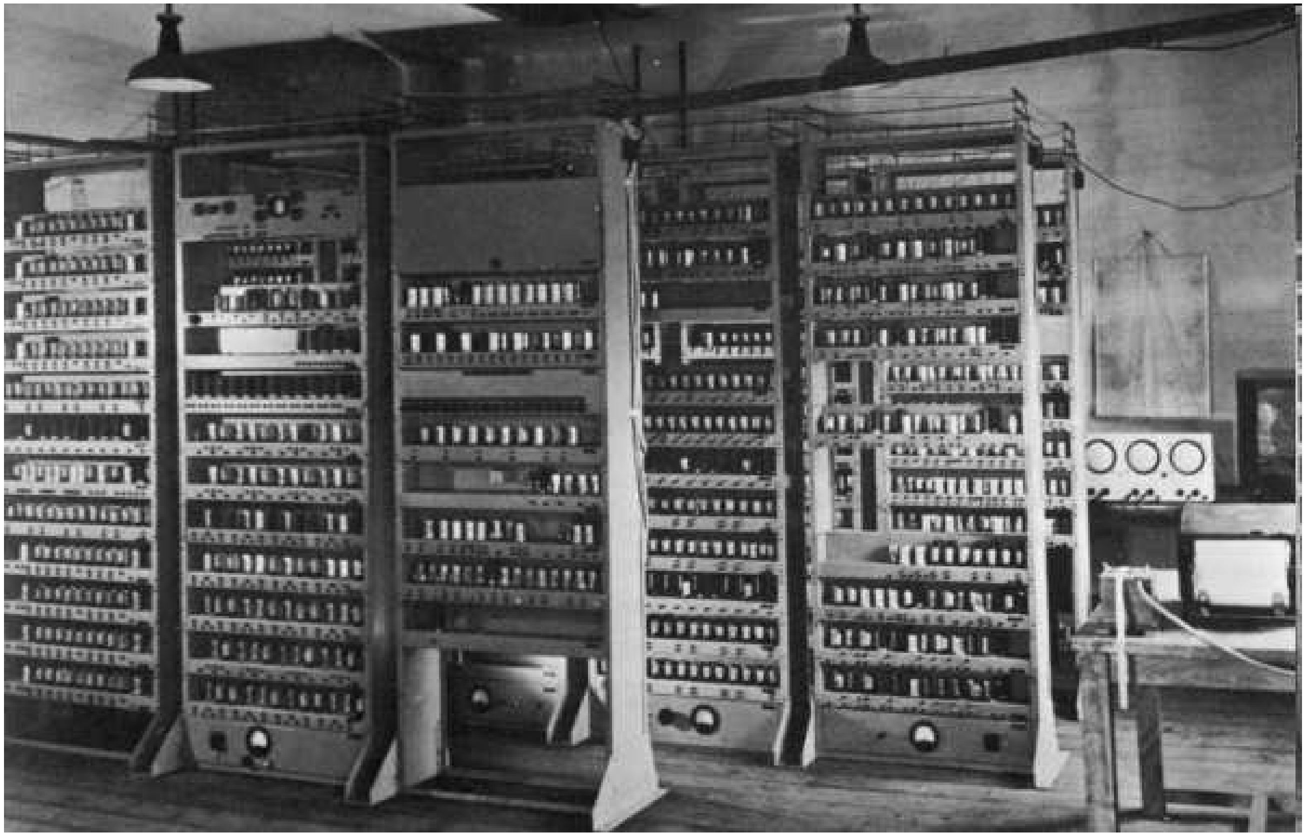
$$x = \frac{2263582143321421502100209233517777}{143560497706190989485475151904721}$$

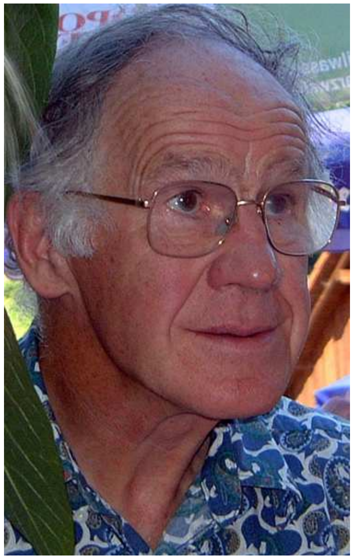
$$y = \frac{186398152584623305624837551485596770028144776655756}{1720094998106353355821008525938727950159777043481}$$

(Found by Michael Stoll in 2002.)

The Central Question

When does an elliptic curve have infinitely many solutions?





Conjectures Proliferated

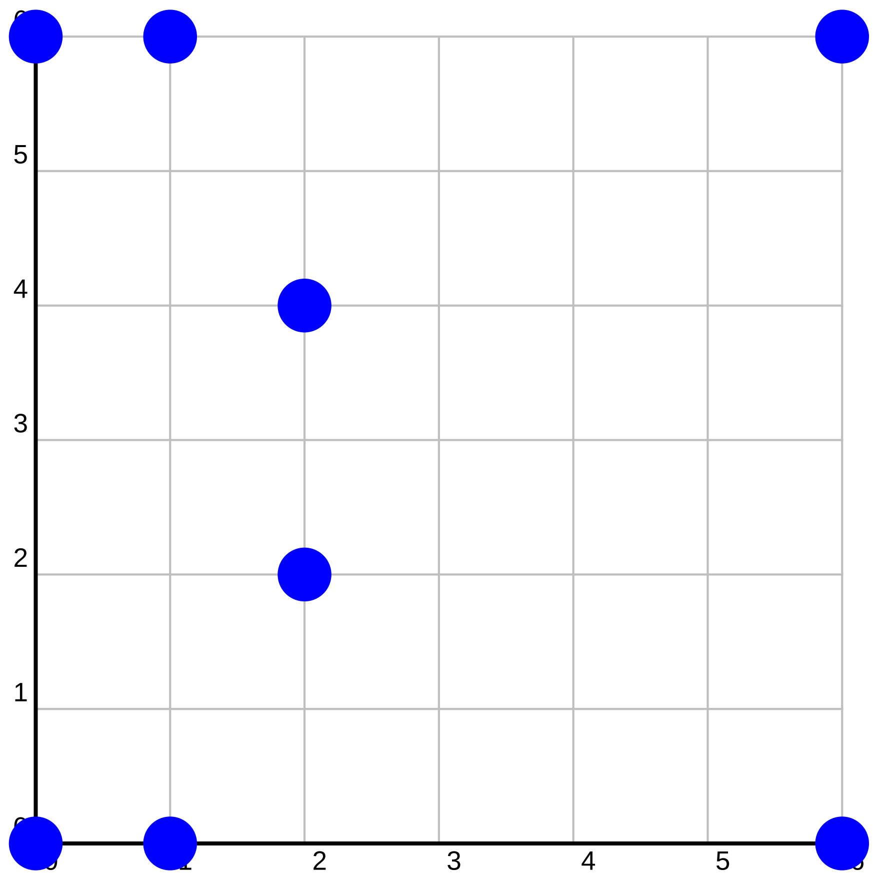
"The subject of this lecture is rather a special one. I want to describe some computations undertaken by myself and Swinnerton-Dyer on EDSAC, by which we have calculated the zeta-functions of certain elliptic curves. As a result of these computations we have found an analogue for an elliptic curve of the Tamagawa number of an algebraic group; and conjectures have proliferated. [...] though the associated theory is both abstract and technically complicated, the objects about which I intend to talk are usually simply defined and often machine computable; **experimentally we have detected certain relations between different invariants**, but we have been unable to approach proofs of these relations, which must lie very deep." – Birch 1965

Counting Solutions Modulo p

$$N(p) = \#\ \text{of solutions} \pmod{p}$$

$$y^2 + y = x^3 - x \pmod{7}$$

∞



$$N(7) = 9$$

t counting gnomes

The Error Term

Let

$$a_p = p + 1 - N(p).$$

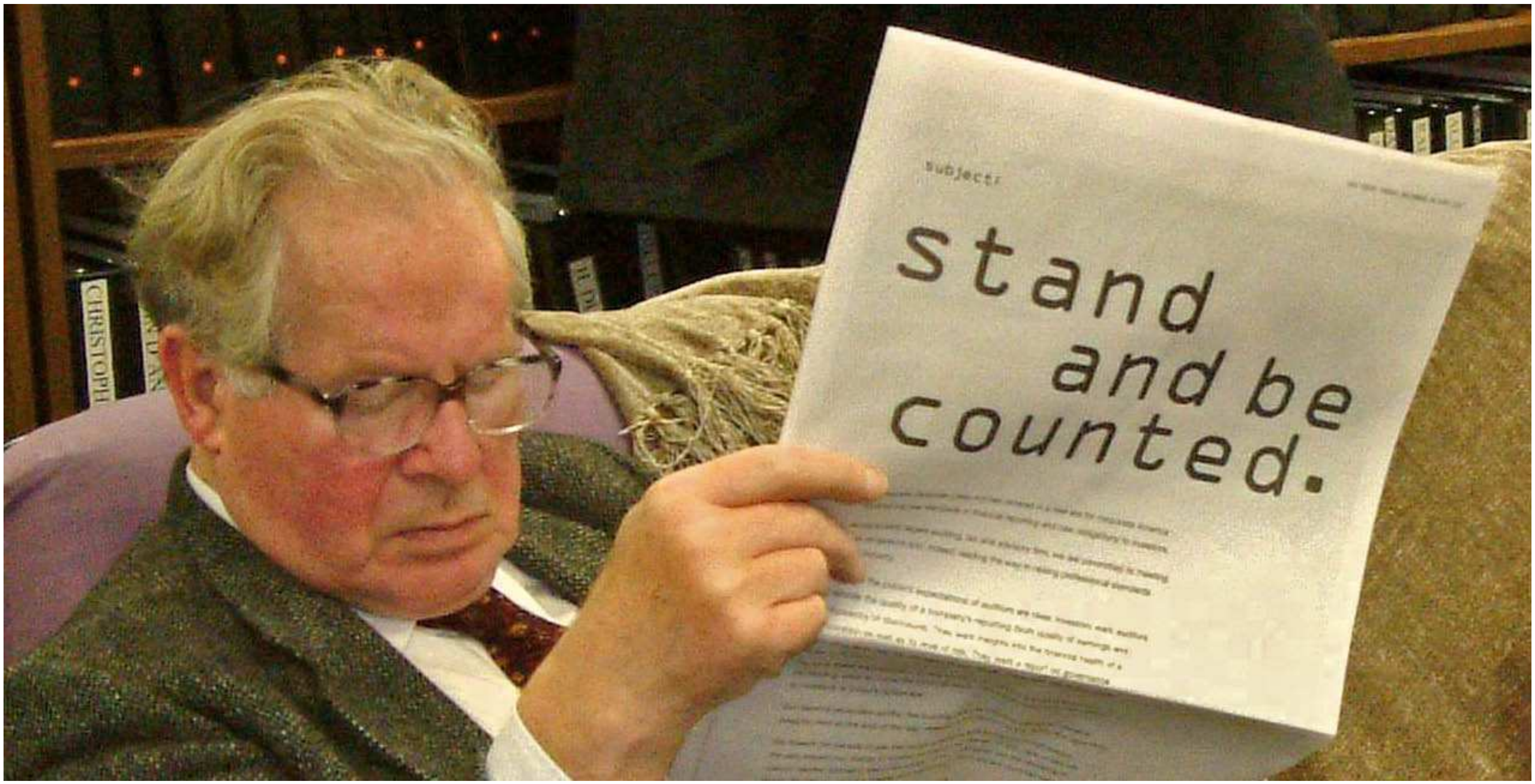
Hasse proved that

$$|a_p| \leq 2\sqrt{p}.$$

$$a_2 = -2, \quad a_3 = -3, \quad a_5 = -2, \quad a_7 = -1, \quad a_{11} = -5, \quad a_{13} = -2,$$

$$a_{17} = 0, \quad a_{19} = 0, \quad a_{23} = 2, \quad a_{29} = 6, \quad \ldots$$

Stand and Be Counted



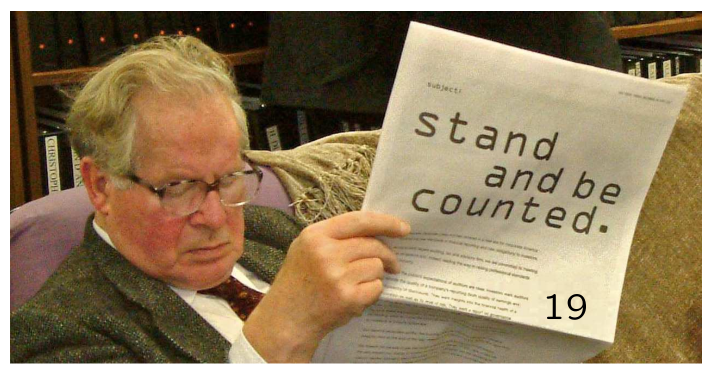
Swinnerton-Dyer

Birch and Swinnerton-Dyer's Guess

If an elliptic curve E has positive rank, then perhaps $N(p)$ is on average larger than p, for many primes p. Maybe

$$f_E(x) = \prod_{p \leq x} \frac{p}{N(p)} \to 0 \textbf{ as } x \to \infty$$

exactly when E **has infinitely many solutions**?



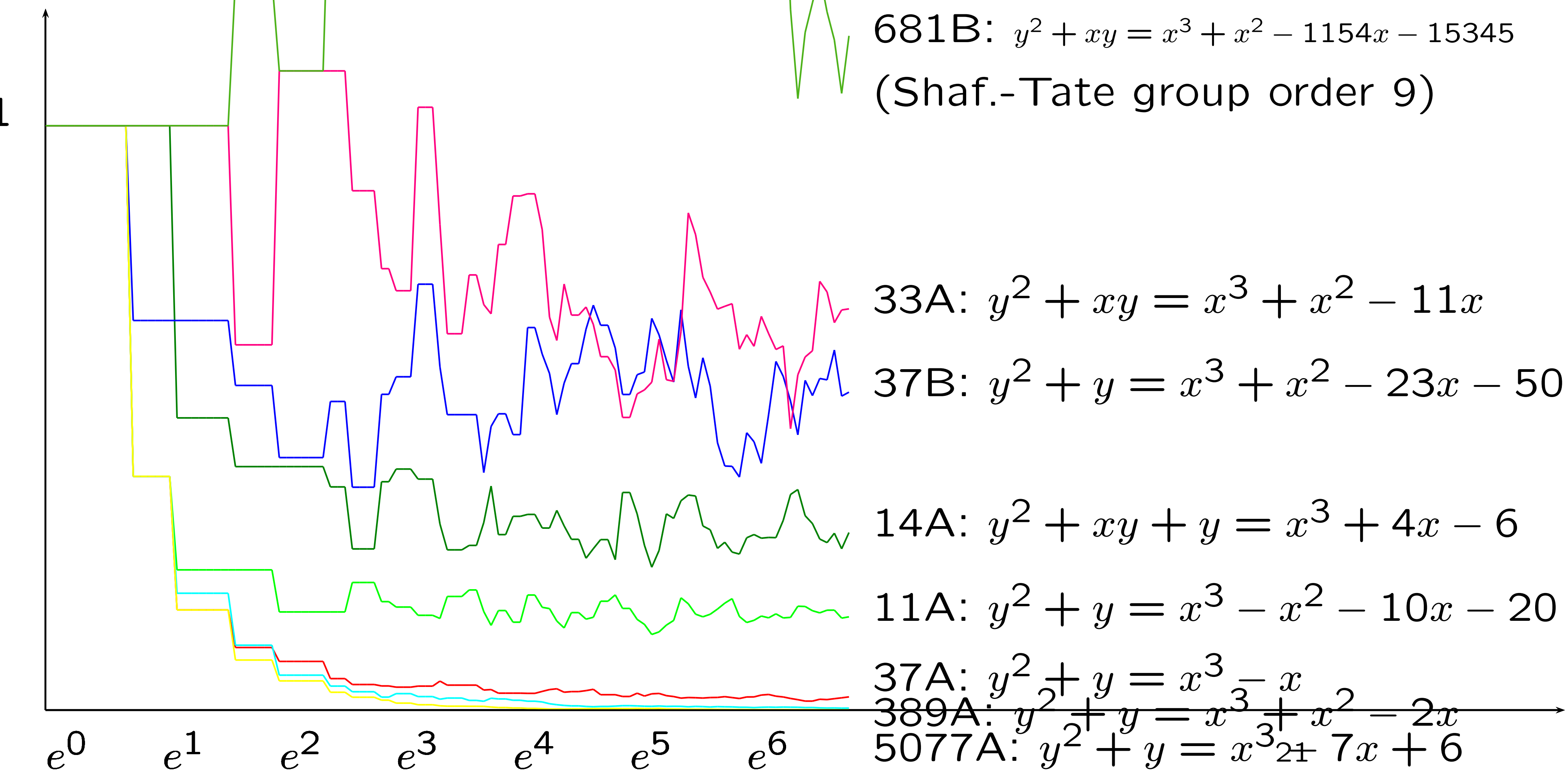
Swinnerton-Dyer

Compute $f_E(x) = \prod_{p \leq x} \frac{p}{N(p)}$

```
sage: E = EllipticCurve([0,0,1,-1,0])
sage: E.Np(7)
9
sage: def f(x): return mul([p / E.Np(p) for p in primes(x)])
   ...:
sage: f(3)
      6/35
sage: f(20)
      2717/69120
sage: f(20)*1.0
      0.0393084490740741
sage: def f(x): return mul([float(p / E.Np(p)) for p in primes(x)])
sage: sage: f(10000)
      0.0126925608355285
sage: f(20000)
      0.0136770159557063
sage: f(100000)
      0.0102764628233953
```

Graphs of $f_E(x) = \Pi_{p\le x} \frac{p}{N(p)}$

The following are log-scale graphs of $f_E(x)$:



Something Better: The L-Function

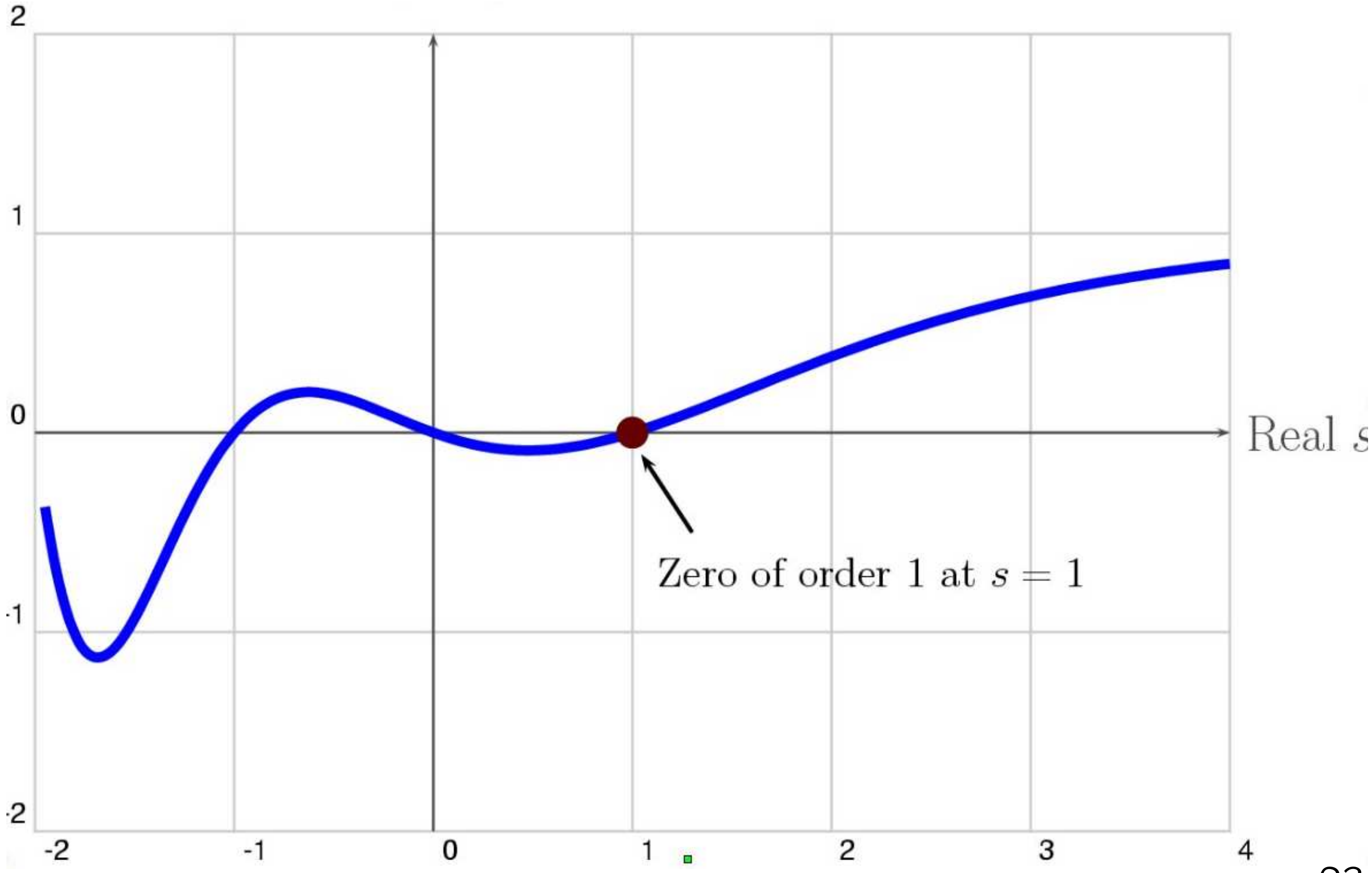
Theorem (Wiles et al., Hecke) This function extends to a holomorphic function on the whole complex plane:

$$L(E,s) = \prod_{p \nmid \Delta} \left(\frac{1}{1 - a_p \cdot p^{-s} + p \cdot p^{-2s}} \right).$$

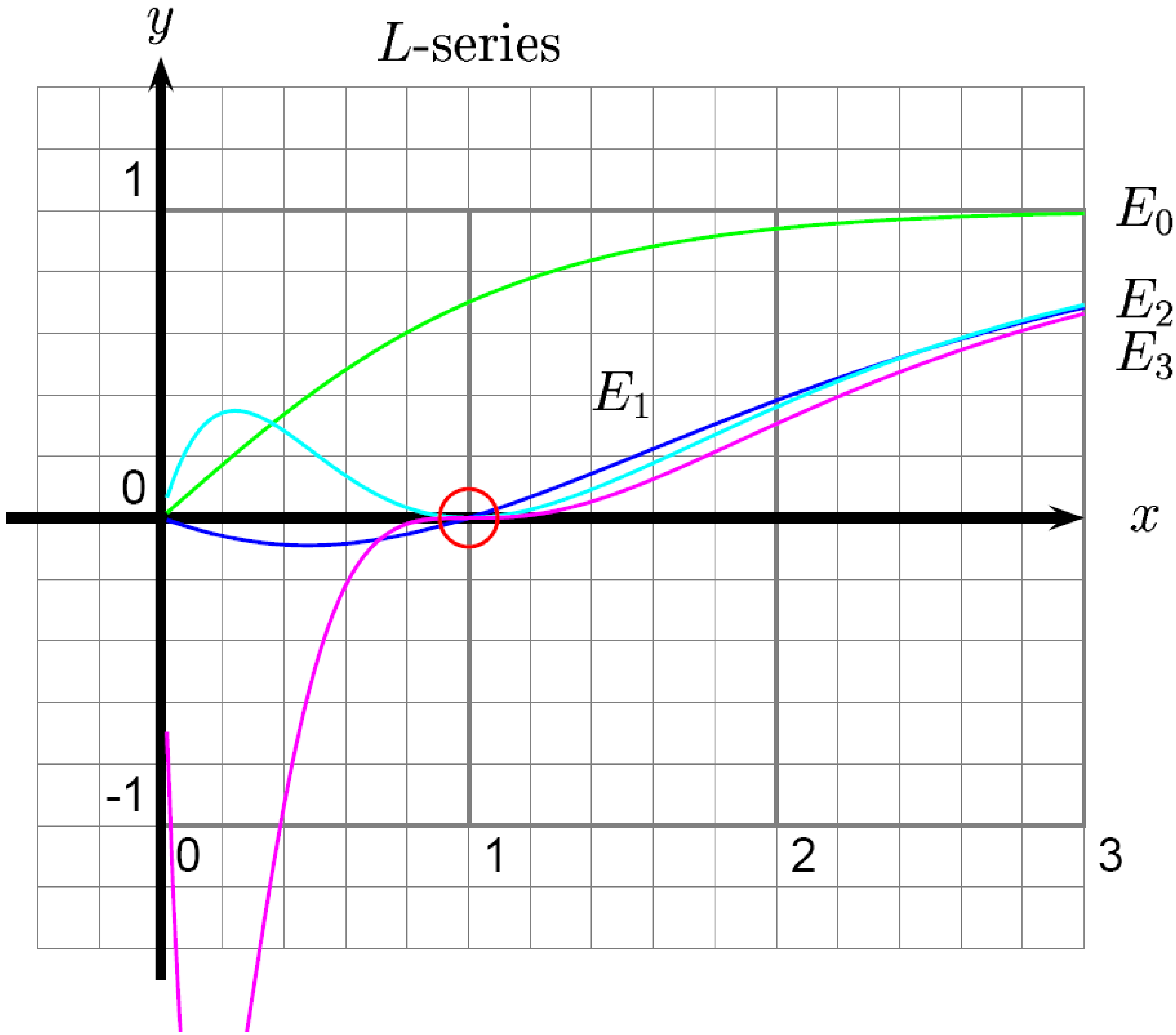
Note that *formally*,

$$L(E,1) = \prod_{p \nmid \Delta} \left(\frac{1}{1 - a_p \cdot p^{-1} + p \cdot p^{-2}} \right) = \prod_{p \nmid \Delta} \left(\frac{p}{p - a_p + 1} \right) = \prod_{p \nmid \Delta} \frac{p}{N_p}$$

Real Graph of the L-Series of $y^2 + y = x^3 - x$



More Graphs of Elliptic Curve L-functions



The Birch and Swinnerton-Dyer Conjecture

Conjecture: Let E be any elliptic curve over $\mathbb{Q}$. *Then E has infinity many solutions if and only if $L(E,1) = 0$.* (More precisely, the order of vanishing of $L(E,s)$ as $s = 1$ equals the rank of $E(\mathbb{Q})$.)

The Kolyvagin and Gross-Zagier Theorem

Theorem: If $L(E,1) \neq 0$ then E has only finitely many solutions.
If $L(E,1) = 0$ but $L'(E,1) \neq 0$, then $E(\mathbb{Q})$ has rank 1.