Chapter 7

Some Differential Equations

Final exam: Wed March 22 7-10pm in Pepper canyon 109.
Today: Section 9.5
Friday: Review (with special guest John Eggers).
Extra Office Hours: Monday 11-2pm

Introduction – not written.

7.1 Separable Equations

A *separable differential equation* is a first order differential equation that can be written in the form

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}.$$

These can be solved by integration, by noting that

$$h(y)dy = f(x)dx,$$

hence

$$\int h(y)dy = \int f(x)dx.$$

This latter equation defines y implicitly as a function of x, and in some cases it is possible to explicitly solve for y as a function of x.

7.2 Logistic Equation

The logistics equation is a differential equation that models population growth. Often in practice a differential equation models some physical situtation, and you should "read it" as doing so.

Exponential growth:

$$\frac{1}{P}\frac{dP}{dt} = k.$$

This says that the "relative (percentage) growth rate" is constant. As we saw before, the solutions are

$$P_{(t)} = P_0 \cdot e^{kt}.$$

Note that this model only works for a little while. In everyday life the growth couldn't actually continue at this rate indefinitely. This exponential growth model ignores limitations on resources, disease, etc. Perhaps there is a better model?

Over time we expect the growth rate should level off, i.e., decrease to 0. What about

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{K}\right), \tag{7.2.1}$$

where K is some large constant called the *carrying capacity*, which is much bigger than $P = P(t)$ at time 0. The carrying capacity is the maximum population that the environment can support. Note that if $P > K$, then $dP/dt < 0$ so the population declines. The differential equation (7.2.1) is called the logistic model (or logistic differential equation). There are, of course, other models one could use, e.g., the Gompertz equation.

First question: are there any *equilibrium solutions* to (7.2.1), i.e., solutions with $dP/dt = 0$, i.e., constant solutions? In order that $dP/dt = 0$ then $0 = k\left(1 - \frac{P}{K}\right)$, so the two equilibrium solutions are $P(t) = 0$ and $P(t) = K$.

The logistic differential equation (7.2.1) is separable, so you can separate the variables with one variable on one side of the equality and one on the other. This means we can easily solve the equation by integrating. We rewrite the equation as

$$\frac{dP}{dt} = -\frac{k}{K}P(P-K).$$

Now separate:

$$\frac{K dP}{P(P-K)} = -k \cdot dt,$$

and integrate both sides

$$\int \frac{K dP}{P(P-K)} = \int -k \cdot dt = -kt + C.$$

On the left side we get

$$\int \frac{K dP}{P(P-K)} = \int \left(\frac{1}{P-K} - \frac{1}{P}\right) dP = \ln|P-K| - \ln|P| + *$$

Thus

$$\ln|K-P| - \ln|P| = -kt + c,$$

so

$$\ln|(K-P)/P| = -kt + c.$$

Now exponentiate both sides:

$$(K-P)/P = e^{-kt+c} = Ae^{-kt}, \qquad \text{where } A = e^c.$$

Thus

$$K = P(1 + Ae^{-kt}),$$

so

$$P(t) = \frac{K}{1 + Ae^{-kt}}.$$

Note that $A = 0$ also makes sense and gives an equilibrium solution. In general we have $\lim_{t\to\infty} P(t) = K$. In any particular case we can determine A as a function of $P_0 = P(0)$ by using that

$$P(0) = \frac{K}{1+A} \qquad \text{so} \qquad A = \frac{K}{P_0} - 1 = \frac{K - P_0}{P_0}.$$