6.5 Power Series

Final exam: Wednesday, March 22, 7-10pm in PCYNH 109. Bring ID!
Quiz 4: This Friday
Today: 11.8 Power Series, 11.9 Functions defined by power series
Next: 11.10 Taylor and Maclaurin series

Recall that a *polynomial* is a function of the form

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k.$$

Polynomials are easy!!!

They are easy to integrate, differentiate, etc.:

$$\frac{d}{dx}\left(\sum_{n=0}^{k} c_n x^n\right) = \sum_{n=1}^{k} n c_n x^{n-1}$$

$$\int \sum_{n=0}^{k} c_n x^n dx = C + \sum_{n=0}^{k} c_n \frac{x^{n+1}}{n+1}.$$

Definition 6.5.1 (Power Series). A *power series* is a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots,$$

where x is a variable and the c_n are coefficients.

A power series is a function of x for those x for which it converges.

Example 6.5.2. Consider

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots.$$

When $|x| < 1$, i.e., $-1 < x < 1$, we have

$$f(x) = \frac{1}{1-x}.$$

But what good could this possibly be? Why is writing the simple function $\frac{1}{1-x}$ as the complicated series $\sum_{n=0}^{\infty} x^n$ of any value?

1. Power series are *relatively easy to work with.* They are "almost" polynomials. E.g.,

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots = \sum_{m=0}^{\infty} (m+1) x^m,$$

where in the last step we "re-indexed" the series. Power series are only "almost" polynomials, since they don't stop; they can go on forever. More precisely, a power series is a limit of polynomials. But in many cases we can treat them like a polynomial. On the other hand, notice that

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_{m=0}^{\infty}(m+1)x^m.$$

2. For many functions, a power series is the *best explicit representation available.*

   Example 6.5.3. Consider $J_0(x)$, the Bessel function of order 0. It arises as a solution to the differential equation $x^2y'' + xy' + x^2y = 0$, and has the following power series expansion:

$$\begin{aligned} J_0(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\ &= 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8 - \frac{1}{14745600}x^{10} + \cdots . \end{aligned}$$

   This series is nice since it converges for all x (one can prove this using the ratio test). It is also one of the most explicit forms of $J_0(x)$.

6.5.1 Shift the Origin

It is often useful to shift the origin of a power series, i.e., consider a power series expanded about a different point.

Definition 6.5.4. The series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a *power series centered at* $x = a$, or "a power series about $x = a$".

Example 6.5.5. Consider

$$\begin{aligned} \sum_{n=0}^{\infty}(x-3)^n &= 1 + (x-3) + (x-3)^2 + \cdots \\ &= \frac{1}{1-(x-3)} \qquad \text{equality valid when } |x-3| < 1 \\ &= \frac{1}{4-x} \end{aligned}$$

Here conceptually we are treating 3 like we treated 0 before.

Power series can be written in different ways, which have different advantages and disadvantages. For example,

$$\begin{aligned} \frac{1}{4-x} &= \frac{1}{4} \cdot \frac{1}{1-x/4} \\ &= \frac{1}{4} \cdot \sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n \qquad \text{converges for all } |x| < 4. \end{aligned}$$

Notice that the second series converges for $|x| < 4$, whereas the first converges only for $|x-3| < 1$, which isn't nearly as good.

6.5.2 Convergence of Power Series

Theorem 6.5.6. *Given a power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$*, there are* exactly *three possibilities:*

1. *The series conveges only when* $x = a$.
2. *The series conveges for all* x.
3. *There is an* $R > 0$ *(called the "radius of convergence") such that* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges for* $|x-a| < R$ *and diverges for* $|x-a| > R$.

Example 6.5.7. For the power series $\sum_{n=0}^{\infty} x^n$, the radius R of convergence is 1.

Definition 6.5.8 (Radius of Convergence). As mentioned in the theorem, R is called the *radius of convergence.*

If the series converges only at $x = a$, we say $R = 0$, and if the series converges everywhere we say that $R = \infty$.

The *interval of convergence* is the set of x for which the series converges. It will be one of the following:

$$(a-R, a+R), \qquad [a-R, a+R), \qquad (a-R, a+R], \qquad [a-R, a+R]$$

The point being that the statement of the theorem only asserts something about convergence of the series on the open interval $(a-R, a+R)$. What happens at the endpoints of the interval is not specified by the theorem; you can only figure it out by looking explicitly at a given series.

Theorem 6.5.9. *If* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is differentiable on* $(a-R, a+R)$*, and*

1. $f'(x) = \sum_{n=1}^{\infty} n \cdot c_n(x-a)^{n-1}$
2. $\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$,

and both the derivative and integral have the same radius of convergence as f.

Example 6.5.10. Find a power series representation for $f(x) = \tan^{-1}(x)$. Notice that

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-1)^n x^{2n},$$

which has radius of convergence $R = 1$, since the above series is valid when $|-x^2| < 1$, i.e., $|x| < 1$. Next integrating, we find that

$$f(x) = c + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1},$$

for some constant c. To find the constant, compute $c = f(0) = \tan^{-1}(0) = 0$. We conclude that

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$