6.3 The Integral and Comparison Tests

Midterm Exam 2: Wednesday March 1 at 7pm in PCYNH 109 (up to *last* lecture)
Today: §7.3–7.4: Integral and comparison tests
Next: §7.6: Absolute convergence; ratio and root tests
Quiz 4 (last quiz): Friday March 10.
Final exam: Wednesday, March 22, 7-10pm in PCYNH 109.

What is $\sum_{n=1}^{\infty} \frac{1}{n^2}$? What is $\sum_{n=1}^{\infty} \frac{1}{n}$?

Recall that Section 6.2 began by asking for the sum of several series. We found the first two sums (which were geometric series) by finding an exact formula for the sum s_N of the first N terms. The third series was

$$A = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots. \tag{6.3.1}$$

It is difficult to find a nice formula for the sum of the first n terms of this series (i.e., I don't know how to do it).

Remark 6.3.1. Since I'm a number theorist, I can't help but make some further remarks about sums of the form (6.3.1). In general, for any $s > 1$ one can consider the sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The number A that we are interested in above is thus $\zeta(2)$. The function $\zeta(s)$ is called the *Riemann zeta function.* There is a natural (but complicated) way of extending $\zeta(s)$ to a (differentiable) function on all complex numbers with a pole at $s = 1$. The *Riemann Hypothesis* asserts that if s is a complex number and $\zeta(s) = 0$ then either s is an even negative integer or $s = \frac{1}{2} + bi$ for some real number b. This is probably *the* most famous unsolved problems in mathematics (e.g., it's one of the Clay Math Institute million dollar prize problems). Another famous open problem is to show that $\zeta(3)$ is not a root of any polynomial with integer coefficients (it is a theorem of Apeéry that $zeta(3)$ is not a fraction).

The function $\zeta(s)$ is incredibly important in mathematics because it governs the properties of prime numbers. The *Euler product* representation of $\zeta(s)$ gives a hint as to why this is the case:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{primes } p} \left(\frac{1}{1 - p^{-s}} \right).$$

To see that this product equality holds when s is real with $\mathrm{Re}(s) > 1$, use Example 6.2.2 with $r = p^{-s}$ and $a = 1$ from the previous lecture. We have

$$\frac{1}{1 - p^{-s}} = 1 + p^{-s} + p^{-2s} + \cdots.$$

Thus

$$\begin{aligned}\prod_{\text{primes } p}\left(\frac{1}{1-p^{-s}}\right) &= \prod_{\text{primes } p}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\\ &= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\cdots\right)\cdot\left(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\cdots\right)\cdots\\ &= \left(1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{n^s},\end{aligned}$$

where the last line uses the distributive law and that integers factor uniquely as a product of primes.

Finally, Figure 6.3.1 is a graph $\zeta(x)$ as a function of a real variable x, and Figure 6.3.2 is a graph of $|\zeta(s)|$ for complex s.

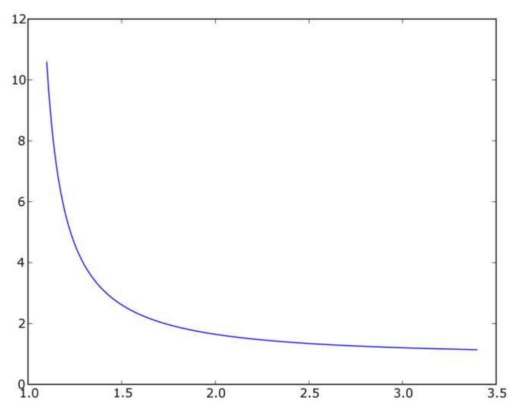

Figure 6.3.1: Riemann Zeta Function: $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$

This section is how to leverage what you've learned so far in this book to say something about sums that are hard (or even "impossibly difficult") to evaluate exactly. For example, notice (by considering a graph of a step function) that if $f(x) = 1/x^2$, then for positive integer t we have

$$\sum_{n=1}^{t}\frac{1}{n^2} \le \frac{1}{1^2} + \int_1^t \frac{1}{x^2}dx.$$

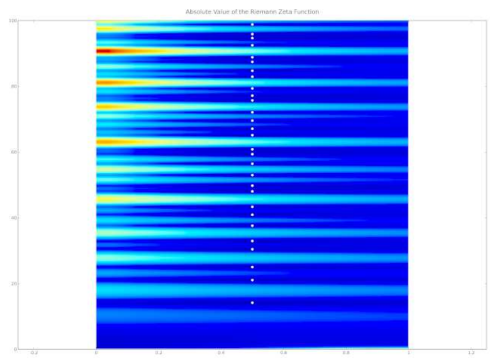

Figure 6.3.2: Absolute Value of Riemann Zeta Function

Thus

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2} &\leq \frac{1}{1^2}+\int_1^{\infty}\frac{1}{x^2}dx\\ &= 1+\lim_{t\to\infty}\int_1^{t}\frac{1}{x^2}dx\\ &= 1+\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t\\ &= 1+\lim_{t\to\infty}\left[-\frac{1}{t}+\frac{1}{1}\right]=2\end{aligned}$$

We conclude that $\sum_{n=1}^{\infty}$ converges, since the sequence of partial sums is getting bigger and bigger and is always ≤ 2. And of course we also know something about $\sum_{n=1}^{\infty}\frac{1}{n^2}$ even though we do not know the exact value: $\sum_{n=1}^{\infty}\frac{1}{n^2}\leq 2$. Using a computer we find that

t	$\sum_{n=1}^{t}\frac{1}{n^2}$
1	1
2	$\frac{5}{4}=1.25$
5	$\frac{5269}{3600}=1.4636\overline{1}$
10	$\frac{1968329}{1270080}=1.54976773117$
100	1.63498390018
1000	1.64393456668
10000	1.64483407185
100000	1.6449240669

The table is consistent with the fact that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges to a number ≤ 2. In fact Euler was the first to compute $\sum_{n=1}^{\infty}$ exactly; he found that the exact value is

$$\frac{\pi^2}{6}=1.644934066848226436472415166646025189218949901206798437735557\ldots$$

There are many proofs of this fact, but they don't belong in this book; you can find them on the internet, and are likely to see one if you take more math classes.

We next consider the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}. \tag{6.3.2}$$

Does it converge? Again by inspecting a graph and viewing an infinite sum as the area under a step function, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &\geq \int_1^{\infty} \frac{1}{x} dx \\ &= \lim_{t\to\infty} [\ln(x)]_1^t \\ &= \lim_{t\to\infty} \ln(t) - 0 = +\infty.\end{aligned}$$

Thus the infinite sum (6.3.2) must also diverge.

We formalize the above two examples as a general test for convergence or divergence of an infinite sum.

Theorem 6.3.2 (Integral Test and Bound). *Suppose $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$ for integers $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ converges. More generally, for any positive integer k,*

$$\int_k^{\infty} f(x)dx \quad \leq \quad \sum_{n=k}^{\infty} a_n \quad \leq \quad a_k + \int_k^{\infty} f(x)dx. \tag{6.3.3}$$

The proposition means that you can determine convergence of an infinite series by determining convergence of a corresponding integral. Thus you can apply the powerful tools you know already for integrals to understanding infinite sums. Also, you can use integration along with computation of the first few terms of a series to approximate a series very precisely.

Remark 6.3.3. Sometimes the first few terms of a series are "funny" or the series doesn't even start at $n = 1$, e.g.,

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^3}.$$

In this case use (6.3.3) with any specific $k > 1$.

Proposition 6.3.4 (Comparison Test). *Suppose $\sum a_n$ and $\sum b_n$ are two series with positive terms. If $\sum b_n$ converges and $a_n \leq b_n$ for all n. then $\sum a_n$ converges. Likewise, if $\sum b_n$ diverges and $a_n \geq b_n$ for all n. then $\sum a_n$ must also diverge.*

Example 6.3.5. *Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge?* No. We have

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{t\to\infty} (2\sqrt{t} - 2\sqrt{1}) = +\infty$$

Example 6.3.6. *Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge?* Let's apply the comparison test: we have $\frac{1}{n^2+1} < \frac{1}{n^2}$ for every n, so

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Alternatively, we can use the integral test, which also gives as a bonus an upper and lower bound on the sum. Let $f(x) = 1/(1+x^2)$. We have

$$\int_1^\infty \frac{1}{1+x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{1+x^2}dx$$
$$= \lim_{t\to\infty}\tan^{-1}(t) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus the sum converges. Moreover, taking $k = 1$ in Theorem 6.3.2 we have

$$\frac{\pi}{4} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \frac{1}{2} + \frac{\pi}{4}.$$

the actual sum is $1.07\ldots$, which is much different than $\sum \frac{1}{n^2} = 1.64\ldots$.

We could prove the following proposition using methods similar to those illustrated in the examples above. Note that this is nicely illustrated in Figure 6.3.1.

Proposition 6.3.7. *The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if* $p > 1$ *and divergent if* $p \leq 1$.

6.3.1 Estimating the Sum of a Series

Suppose $\sum a_n$ is a convergent sequence of positive integers. Let

$$R_m = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{m} a_n = \sum_{n=m+1}^{\infty} a_m$$

which is the error if you approximate $\sum a_n$ using the first n terms. From Theorem 6.3.2 we get the following.

Proposition 6.3.8 (Remainder Bound). *Suppose* f *is a continuous, positive, decreasing function on* $[m,\infty)$ *and* $\sum a_n$ *is convergent. Then*

$$\int_{m+1}^{\infty} f(x)dx \leq R_m \leq \int_m^{\infty} f(x)dx.$$

Proof. In Theorem 6.3.2 set $k = m + 1$. That gives

$$\int_{m+1}^{\infty} f(x)dx \quad\leq\quad \sum_{n=m+1}^{\infty} a_n \quad\leq\quad a_{m+1} + \int_{m+1}^{\infty} f(x)dx.$$

But

$$a_{m+1} + \int_{m+1}^{\infty} f(x)dx \leq \int_m^{\infty} f(x)dx$$

since f is decreasing and $f(m+1) = a_{m+1}$. □

Example 6.3.9. Estimate $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ using the first 10 terms of the series. We have

$$\sum_{n=1}^{10} = \frac{19164113947}{16003008000} = 1.197531985674193\ldots$$

The proposition above with $m = 10$ tells us that

$$0.00413223140495867\ldots = \int_{11}^{\infty} \frac{1}{x^3}dx \leq \zeta(3) - \sum_{n=1}^{10} \leq \int_{10}^{\infty} \frac{1}{x^3}dx = \frac{1}{2 \cdot 10^2} = \frac{1}{200} = 0.005.$$

In fact,

$$\zeta(3) = 1.202056903159594285399738161511449990\ldots$$

and we hvae

$$\zeta(3) - \sum_{n=1}^{10} = 0.0045249174854010\ldots,$$

so the integral error bound was really good in this case.

Example 6.3.10. Determine if $\sum_{n=1}^{\infty} \frac{2006}{117n^2+41n+3}$ convergers or diverges. Answer: It converges, since

$$\frac{2006}{117n^2 + 41n + 3} \leq \frac{2006}{117n^2} = \frac{2006}{117} \cdot \frac{1}{n^2},$$

and $\sum \frac{1}{n^2}$ converges.