5.4 Factoring Polynomials

Quizes today!

How do you compute something like

$$\int \frac{x^2+2}{(x-1)(x+2)(x+3)}dx?$$

So far you have no method for doing this. The trick (which is called partial fraction decomposition), is to write

$$\int \frac{x^2+2}{x^3+4x^2+x-6}dx = \int \frac{1}{4(x-1)} - \frac{2}{x+2} + \frac{11}{4(x+3)}dx \tag{5.4.1}$$

The integral on the right is then easy to do (the answer involves ln's).

But *how on earth* do you right the rational function on the left hand side as a sum of the nice terms of the right hand side? Doing this is called "partial fraction decomposition", and it is a fundamental idea in mathematics. It relies on our ability to factor polynomials and saolve linear equations. As a first hint, notice that

$$x^3+4x^2+x-6 = (x-1)\cdot(x+2)\cdot(x+3),$$

so the denominators in the decomposition correspond to the factors of the denominator.

Before describing the secret behind (5.4.1), we'll discuss some background about how polynomials and rational functions work.

Theorem 5.4.1 (Fundamental Theorem of Algebra). *If $f(x) = a_n x^n + \cdots a_1 x + a_0$ is a polynomial, then there are complex numbers $c, \alpha_1, \ldots \alpha_n$ such that*

$$f(x) = c(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n).$$

Example 5.4.2. For example,

$$3x^2+2x-1 = 3\cdot\left(x-\frac{1}{3}\right)\cdot(x+1).$$

And

$$(x^2+1) = (x+i)^2\cdot(x-i)^2.$$

If $f(x)$ is a polynomial, the roots α of f correspond to the factors of f. Thus if

$$f(x) = c(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

then $f(\alpha_i) = 0$ for each i (and nowhere else).

Definition 5.4.3 (Multiplicity of Zero). The *multiplicity of a zero* α of $f(x)$ is the number of times that $(x-\alpha)$ appears as a factor of f.

For example, if $f(x) = 7(x-2)^{99}\cdot(x+17)^5\cdot(x-\pi)^2$, then 2 is a zero with multiplicity 99, π is a zero with multiplicity 2, and -1 is a "zero multiplicity 0".

Definition 5.4.4 (Rational Function). A *rational function* is a quotient

$$f(x) = \frac{g(x)}{h(x)},$$

where $g(x)$ and $h(x)$ are polynomials.

For example,

$$f(x) = \frac{x^{10}}{(x-i)^2(x+\pi)(x-3)^3} \tag{5.4.2}$$

is a rational function.

Definition 5.4.5 (Pole). A *pole* of a rational function $f(x)$ is a complex number α such that $|f(x)|$ is unbounded as $x \to \alpha$.

For example, for (5.4.2) the poles are at i, π, and 3. They have multiplicity 2, 1, and 3, respectively.