5.2 Trigonometric Integrals

Friday: Quiz 2
Next: Trig subst.

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \text{and} \qquad \sin^2(x) = \frac{1-\sin(2x)}{2}. \tag{5.2.1}$$

Example 5.2.1. Compute $\int \sin^3(x)dx$.
We use trig. identities and compute the integral directly as follows:

$$\begin{aligned}
\int \sin^3(x)dx &= \int \sin^2(x)\sin(x)dx \\
&= \int [1-\cos^2(x)]\sin(x)dx \\
&= -\cos(x) + \frac{1}{3}\cos^3(x) + c \qquad \text{(substitution } u = \cos(x))
\end{aligned}$$

This always works for odd powers of $\sin(x)$.

Example 5.2.2. What about *even* powers?! Compute $\int \sin^4(x)dx$. We have

$$\begin{aligned}
\sin^4(x) &= [\sin^2(x)]^2 \\
&= \left[\frac{1-\cos(2x)}{2}\right]^2 \\
&= \frac{1}{4}\cdot\left[1 - 2\cos(2x) + \cos^2(2x)\right] \\
&= \frac{1}{4}\left[1 - 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\right]
\end{aligned}$$

Thus

$$\begin{aligned}
\int \sin^4(x)dx &= \int \left[\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right]dx \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c.
\end{aligned}$$

Key Trick: Realize that we should write $\sin^4(x)$ as $(\sin^2(x))^2$. The rest is straightforward.

Example 5.2.3. This example illustrates a method for computing integrals of trig functions that doesn't require knowing any trig identities at all or any tricks. It is very tedious though. We compute $\int \sin^3(x)dx$ using *complex exponentials*. We have

$$\cos(x) = \frac{e^{ix}+e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}.$$

hence

$$\begin{aligned}
\int \sin^3(x)dx &= \int \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3 dx \\
&= -\frac{1}{8i}\int (e^{ix}-e^{-ix})^3 dx \\
&= -\frac{1}{8i}\int (e^{ix}-e^{-ix})(e^{ix}-e^{-ix})(e^{ix}-e^{-ix})dx \\
&= -\frac{1}{8i}\int (e^{2ix}-2+e^{-2ix})(e^{ix}-e^{-ix})dx \\
&= -\frac{1}{8i}\int e^{3ix}-e^{ix}-2e^{ix}+2e^{-ix}+e^{-ix}-e^{-3ix}dx \\
&= -\frac{1}{8i}\int e^{3ix}-e^{-3ix}+3e^{-ix}-3e^{ix}dx \\
&= -\frac{1}{8i}\left(\frac{e^{3ix}}{3i}-\frac{e^{-3ix}}{-3i}+\frac{3e^{-ix}}{-i}-\frac{3e^{ix}}{i}\right)+c \\
&= \frac{1}{4}\left(\frac{1}{3}\cos(3x)-3\cos(x)\right)+c \\
&= \frac{1}{12}\cos(3x)-\frac{3}{4}\cos(x)+c
\end{aligned}$$

The answer looks totally different, but is in fact the same function.

Here are some more identities that we'll use in illustrating some tricks below.

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

and

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$

Also,

$$1+\tan^2(x) = \sec^2(x).$$

Example 5.2.4. Compute $\int \tan^3(x)dx$. We have

$$\begin{aligned}
\int \tan^3(x)dx &= \int \tan(x)\tan^2(x)dx \\
&= \int \tan(x)\left[\sec^2(x)-1\right]dx \\
&= \int \tan(x)\sec^2(x)dx - \int \tan(x)dx \\
&= \frac{1}{2}\tan^2(x) - \ln|\sec(x)| + c
\end{aligned}$$

Here we used the substitution $u = \tan(x)$, so $du = \sec^2(x)dx$, so

$$\int \tan(x)\sec^2(x)dx = \int udu = \frac{1}{2}u^2 + c = \frac{1}{2}\tan^2(x) + c.$$

Also, with the substitution $u = \cos(x)$ and $du = -\sin(x)dx$ we get

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{1}{u}du = -\ln|u| + c = -\ln|\sec(x)| + c.$$

Key trick: Write $\tan^3(x)$ as $\tan(x)\tan^2(x)$.

Example 5.2.5. Here's one that combines trig identities with the funnest variant of integration by parts. *Compute* $\int \sec^3(x)dx$.
We have
$$\int \sec^3(x)dx = \int \sec(x)\sec^2(x)dx.$$
Let's use integration by parts.
$$\begin{aligned} u &= \sec(x) & v &= \tan(x) \\ du &= \sec(x)\tan(x)dx & dv &= \sec^2(x)dx \end{aligned}$$
The above integral becomes
$$\begin{aligned} \int \sec(x)\sec^2(x)dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)dx \\ &= \sec(x)\tan(x) - \int \sec(x)[\sec^2(x)-1]dx \\ &= \sec(x)\tan(x) - \int \sec^3(x) + \int \sec(x)dx \\ &= \sec(x)\tan(x) - \int \sec^3(x) + \ln|\sec(x)+\tan(x)| \end{aligned}$$

This is familiar. Solve for $\int \sec^3(x)$. We get
$$\int \sec^3(x)dx = \frac{1}{2}\Big[\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\Big] + c$$

5.2.1 Some Remarks on Using Complex-Valued Functions

Consider functions of the form
$$f(x) + ig(x), \tag{5.2.2}$$
where x is a real variable and f, g are real-valued functions. For example,
$$e^{ix} = \cos(x) + i\sin(x).$$
We observed before that
$$\frac{d}{dx}e^{wx} = we^{wx}$$
hence
$$\int e^{wx}dx = \frac{1}{w}e^{wx} + c.$$
For example, writing it e^{ix} as in (5.2.2), we have
$$\begin{aligned} \int e^{ix}dx &= \int \cos(x)dx + i\int \sin(x)dx \\ &= \sin(x) - i\cos(x) + c \\ &= -i(\cos(x) + i\sin(x)) + c \\ &= \frac{1}{i}e^{ix}. \end{aligned}$$

Example 5.2.6. Let's compute $\int \frac{1}{x+i}dx$. Wouldn't it be nice if we could just write $\ln(x+i)+c$? This is useless for us though, since we haven't even *defined* $\ln(x+i)$! However, we can "rationalize the denominator" by writing

$$\begin{aligned}\int \frac{1}{x+i}dx &= \int \frac{1}{x+i}\cdot\frac{x-i}{x-i}dx \\ &= \int \frac{x-i}{x^2+1}dx \\ &= \int \frac{x}{x^2+1}dx - i\int \frac{1}{x^2+1}dx \\ &= \frac{1}{2}\ln|x^2+1| - i\tan^{-1}(x)+c\end{aligned}$$

This informs how we would define $\ln(z)$ for z complex (which you'll do if you take a course in complex analysis). **Key trick:** Get the i in the numerator.

The next example illustrates an alternative to the method of Section 5.2.

Example 5.2.7.

$$\begin{aligned}\int \sin(5x)\cos(3x)dx &= \int \left(\frac{e^{i5x}-e^{-i5x}}{2i}\right)\left(\cdot\frac{e^{i5x}+e^{-i5x}}{2}\right)dx \\ &= \frac{1}{4i}\int \left(e^{i8x}-e^{-i8x}+e^{i2x}-e^{-i2x}\right)dx+c \\ &= \frac{1}{4i}\left(\frac{e^{i8x}}{8i}+\frac{e^{-i8x}}{8i}+\frac{e^{i2x}}{2i}+\frac{e^{-i2x}}{2i}\right)+c \\ &= -\frac{1}{4}\left[\frac{1}{4}\cos(8x)+\cos(2x)\right]+c\end{aligned}$$

This *is* more tedious than the method in 5.2. But it is *completely straightforward*. You don't need any trig formulas or anything else. You just multiply it out, integrate, etc., and remember that $i^2=-1$.