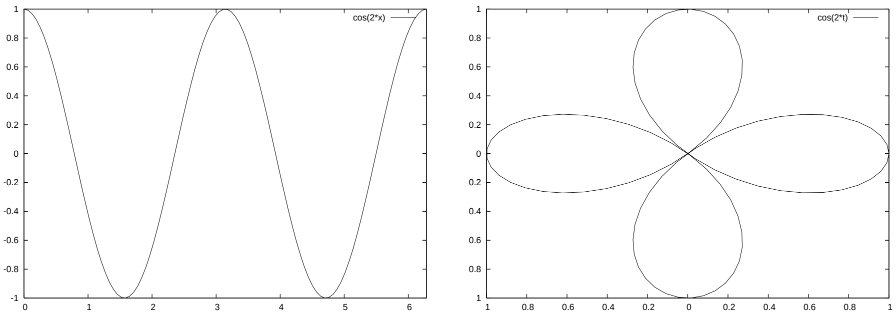
2.5 Areas in Polar Coordinates

Exam 1 Wed Feb 1 7:00pm in Pepper Canyon 109 (not 106!! different class there!)
Office hours: 2:45pm–4:15pm
Next: Complex numbers (appendix G); complex exponentials (supplement, which is freely available online).
We will *not* do arc length.

People were most confused last time by plotting curves in polar coordinates. (1) it *is* tedious, but easier if you do a few and know what they look like (just plot some points and see); there's not much to it, except plug in values and see what you get, and (2) can sometimes convert to a curve in (x, y) coordinates, which might be easier.
GOAL for today: Integration in the context of polar coordinates. Get much better at working with polar coordinates!

Example 2.5.1. (From Stewart.) Find the area enclosed by one leaf of the four-leaved rose $r = \cos(2\theta)$. To find the area using the methods we know so far, we would need

Figure 2.5.1: Graph of $y = \cos(2x)$ and $r = \cos(2\theta)$



to find a function $y = f(x)$ that gives the height of the leaf.

Multiplying both sides of the equation $r = \cos(2\theta)$ by r yields

$$r^2 = r\cos(2\theta) = r(\cos^2\theta - \sin^2\theta) = \frac{1}{r}((r\cos\theta)^2 - (r\sin\theta)^2).$$

Because $r^2 = x^2 + y^2$ and $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$x^2 + y^2 = \frac{1}{\sqrt{x^2 + y^2}}(x^2 - y^2).$$

Solving for y is a crazy mess, and then integrating? It seems impossible!

But it isn't... if we remember the basic idea of calculus: subdivide and take a limit.

[[Draw a section of a curve $r = f(\theta)$ for θ in some interval $[a, b]$, and shade in the area of the arc.]]

Remark 2.5.2. We will almost *never* talk about angles in degrees—we'll almost always use radians.

We know how to compute the area of a sector, i.e., piece of a circle with angle θ. [[draw picture]]. This is the basic polar region. The area is

$$A = (\text{fraction of the circle}) \cdot (\text{area of circle}) = \left(\frac{\theta}{2\pi}\right) \cdot \pi r^2 = \frac{1}{2} r^2 \theta.$$

We now imitate what we did before with Riemann sums. We chop up, approximate, and take a limit. Break the interval of angles from a to b into n subintervals. Choose θ_i^* in each interval. The area of each slice is approximately $(1/2) f(\theta_i^*)^2 \theta_i^2$. Thus

$$A = \text{Area of the shaded region} \sim \sum_{i=1}^{n} \frac{1}{2} f(\theta_i^*)^2 \Delta(\theta).$$

Taking the limit, we see that

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} f(\theta_i^*)^2 \Delta(\theta) = \frac{1}{2} \cdot \int_a^b f(\theta)^2 d\theta.$$

Amazing! By understanding the definition of Riemann sum, we've derived a formula for areas swept out by a polar graph. But does it work in practice? Let's revisit our clover leaf.

2.5.1 Examples

Example 2.5.3. Find the area enclosed by one leaf of the four-leaved rose $r = \cos(2\theta)$. Solution: We need the boundaries of integration. Start at $\theta = -\pi/4$ and go to $\theta = \pi/4$. As a check, note that $\cos((-\pi/4) \cdot 2) = 0 = \cos((\pi/4) \cdot 2)$. We evaluate

$$\begin{aligned}
\frac{1}{2} \cdot \int_{-\pi/4}^{\pi/4} \cos(2\theta)^2 d\theta &= \int_0^{\pi/4} \cos(2\theta)^2 d\theta \qquad \text{(even function)} \\
&= \frac{1}{2} \int_0^{\pi/4} (1 + \cos(4\theta)) d\theta \\
&= \frac{1}{2} \left[\theta + \frac{1}{4} \cdot \sin(4\theta)\right]_0^{\pi/4} \\
&= \frac{\pi}{8}.
\end{aligned}$$

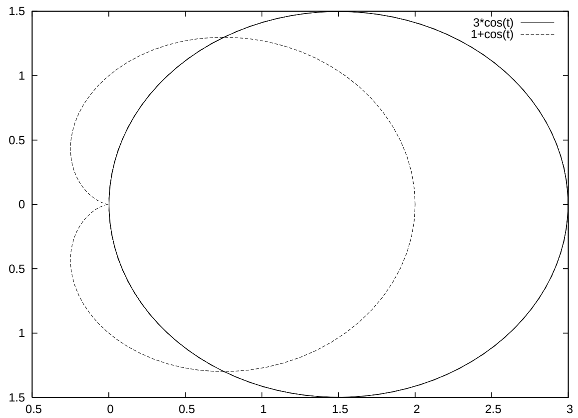
We used that $\cos^2(x) = (1 + \cos(2x))/2$ and $\sin^2(x) = (1 - \sin(2x))/2$, which follow from

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x).$$

Example 2.5.4. Find area of region inside the curve $r = 3\cos(\theta)$ and outside the cardiod curve $r = 1 + \cos(\theta)$.
Solution: This is the same as before. It's the difference of two areas. Figure out the limits, which are where the curves intersect, i.e., the θ such that

$$3\cos(\theta) = 1 + \cos(\theta).$$

Figure 2.5.2: Graph of $r = 3\cos(\theta)$ and $r = 1 + \cos(\theta)$



Solving, $2\cos(\theta) = 1$, so $\cos(\theta) = 1/2$, hence $\theta = \pi/3$ and $\theta = -\pi/3$. Thus the area is

$$
\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (3\cos(\theta))^2 - (1+\cos(\theta))^2 d\theta \\
&= \int_0^{\pi/3} (3\cos(\theta))^2 - (1+\cos(\theta))^2 d\theta \qquad \text{(even function)} \\
&= \int_0^{\pi/3} (8\cos^2(\theta) - 2\cos(\theta) - 1) d\theta \\
&= \int_0^{\pi/3} \left(8 \cdot \frac{1}{2}(1+\cos(2\theta)) - 2\cos(\theta) - 1\right) d\theta \\
&= \int_0^{\pi/3} 3 + 4\cos(2\theta) - 2\cos(\theta) d\theta \\
&= \Big[3\theta + 2\sin(2\theta) - 2\sin(\theta)\Big]_0^{\pi/3} \\
&= \pi + 2 \cdot \sqrt{\frac{3}{2}} - 2\sqrt{\frac{3}{2}} - 0 - 2\cdot 0 - 2 \cdot 0 \\
&= \pi
\end{aligned}
$$