Math 129: Algebraic Number Theory

Lecture 22: Properties of the Adeles

William Stein (based closely on Cassels's *Global Fields* article in Cassels-Fröhlich)

Thursday, April 29, 2004

The final project and the short take-home final will both be due on Monday, May 24, at NOON. You should turn your final projects in to me **via email**, since I will be posting all final projects on a web page. Do a good job on your final project, since a few years from now it will probably still be on that web page :-).

You should not worry too much about the final exam, which is worth only 20% of your grade. It will be take home, and you will be able to use all resources except communication with other people. Thus you can search in the library and online for ideas and answer. It will not be something you can really study for, and will probably take several hours.

What is a "final project"? How long should it be? This depends on you. Your goal for the final project should be to write something that you think other people in the course or people browsing the Math 129 web page would find an interesting and reasonably accessible read. Explore a topic, try to understand something interesting about it, and write it up so that somebody with an advanced undergraduate background in mathematics can understand it. A bad project would be one with nothing "interesting" in it that is riddled with errors. In contrast, a good project would be something that people with some interest in algebraic number theory would *actually want to read*, and is technically correct. Don't aim so much to impress me with how much you were able to learn, but aim to produce a document that will add some variety to the Math 129 web page.

---

14 The Adele Ring

Let K be a global field. For each normalization $|\cdot|_v$ of K, let K_v denote the completion of K. If $|\cdot|_v$ is non-archimedean, let $\mathcal{O}_v$ denote the ring of integers of K_v.

Definition 14.1 (Adele Ring). The *adele ring* $\mathbb{A}_K$ of K is the topological ring whose underlying topological space is the restricted topological product of the K_v with respect to the $\mathcal{O}_v$, and where addition and multiplication are defined componentwise:

$$(\mathbf{x}\mathbf{y})_v = \mathbf{x}_v\mathbf{y}_v \qquad (\mathbf{x}+\mathbf{y})_v = \mathbf{x}_v + \mathbf{y}_v \qquad \text{for } \mathbf{x},\mathbf{y} \in \mathbb{A}_K. \tag{14.1}$$

It is readily verified that (i) this definition makes sense, i.e., if $\mathbf{x},\mathbf{y} \in \mathbb{A}_K$, then $\mathbf{x}\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$, whose components are given by (14.1), are also in $\mathbb{A}_K$, and (ii) that addition and multiplication are continuous in the $\mathbb{A}_K$-topology, so $\mathbb{A}_K$ is a topological ring, as asserted. Also, $\mathbb{A}_K$ is locally compact because the K_v are locally compact, and the $\mathcal{O}_v$ are compact.

There is a natural continuous ring inclusion

$$K \hookrightarrow \mathbb{A}_K \tag{14.2}$$

that sends $x \in K$ to the adele every one of whose components is x. This is an adele because $x \in \mathcal{O}_v$ for almost all v, by the lemma we proved last time. . The map is injective because each map $K \to K_v$ is an inclusion.

Definition 14.2 (Principal Adeles). The image of (14.2) is the ring of *principal adeles.*

It will cause no trouble to identify K with the principal adeles, so we shall speak of K as a subring of $\mathbb{A}_K$.

Formation of the adeles is compatibility with base change, in the following sense.

Lemma 14.3. *Suppose L is a finite (separable) extension of the global field K. Then*

$$\mathbb{A}_K \otimes_K L \cong \mathbb{A}_L \tag{14.3}$$

both algebraically and topologically. Under this isomorphism, $L \cong K \otimes_K L \subset \mathbb{A}_K \otimes_K L$ maps isomorphically onto $L \subset \mathbb{A}_L$.

Proof. We first establish an isomorphism of the two sides of (14.3) as topological spaces. Let $\omega_1, \ldots, \omega_n$ be a basis for L/K and let v run through the normalized valuations on K. The left hand side of (14.3), with the tensor product topology, is the restricted product of the tensor products

$$K_v \otimes_K L \cong K_v \cdot \omega_1 \oplus \cdots \oplus K_v \cdot \omega_n$$

with respect to the integers

$$\mathcal{O}_v \cdot \omega_1 \oplus \cdots \oplus \mathcal{O}_v \cdot \omega_n. \tag{14.4}$$

(An element of the left hand side is a finite linear combination $\sum \mathbf{x}_i \otimes a_i$ of adeles $\mathbf{x}_i \in \mathbb{A}_K$ and coefficients $a_i \in L$, and there is a natural isomorphism from the ring of such formal sums to the restricted product of the $K_v \otimes_K L$.)

We proved before that

$$K_v \otimes_K L \cong L_{w_1} \oplus \cdots \oplus L_{w_g},$$

where $w_1, \ldots, w_g$ are the normalizations of the extensions of v to L. Furthermore, as we proved using discriminants, the above identification identifies (14.4) with

$$\mathcal{O}_{L_{w_1}} \oplus \cdots \oplus \mathcal{O}_{L_{w_g}},$$

for almost all v. The left hand side of (14.3) is the restricted product of the $L_{w_1} \oplus \cdots \oplus L_{w_g}$ with respect to the $\mathcal{O}_{L_{w_1}} \oplus \cdots \oplus \mathcal{O}_{L_{w_g}}$. This is canonically isomorphic to the restricted product of all completions L_w with respect to $\mathcal{O}_w$, which is the right hand side of (14.3). This establishes an isomorphism between the two sides of (14.3) as topological spaces. The map is also a ring homomorphism, so the two sides are algebraically isomorphic, as claimed. $\square$

Corollary 14.4. *Let $\mathbb{A}_K^+$ denote the topological group obtained from the additive structure on $\mathbb{A}_K$. Suppose L is a finite seperable extension of K. Then*

$$\mathbb{A}_L^+ = \mathbb{A}_K^+ \oplus \cdots \oplus \mathbb{A}_K^+, \qquad ([L : K] \text{ summands}).$$

In this isomorphism the additive group $L^+ \subset \mathbb{A}_L^+$ of the principal adeles is mapped into $K^+ \oplus \cdots \oplus K^+$.

Proof. For any nonzero $\omega \in L$, the subgroup $\omega \cdot \mathbb{A}_K^+$ of $\mathbb{A}_L^+$ is isomorphic as a topological group to $\mathbb{A}_K^+$ (the isomorphism is multiplication by $1/\omega$). By Lemma 14.3, we have isomorphisms

$$\mathbb{A}_L^+ = \mathbb{A}_K^+ \otimes_K L \cong \omega_1 \cdot \mathbb{A}_K^+ \oplus \cdots \oplus \omega_n \cdot \mathbb{A}_K^+ \cong \mathbb{A}_K^+ \oplus \cdots \oplus \mathbb{A}_K^+.$$

If $a \in L$, write $a = \sum b_i \omega_i$, with $b_i \in K$. Then a maps via the above map to

$$x = (\omega_1(b_1), \ldots, \omega_n(b_n)),$$

where (b_i) denotes the adele defined by b_i. Under the final map, x maps to the tuple

$$(b_1, \ldots, b_n) \in K \oplus \cdots \oplus K \subset \mathbb{A}_K^+ \oplus \cdots \oplus \mathbb{A}_K^+.$$

This proves the second claim of the corollary. □

Theorem 14.5. *The global field K is discrete in $\mathbb{A}_K$ and the quotient $\mathbb{A}_K^+/K^+$ of additive groups is compact in the quotient topology.*

At this point Cassels remarks

> "It is impossible to conceive of any other uniquely defined topology on K. This metamathematical reason is more persuasive than the argument that follows!"

Proof. Corollary 14.4, with K for L and $\mathbf{Q}$ or $\mathbf{F}(t)$ for K, shows that it is enough to verify the theorem for $\mathbf{Q}$ or $\mathbf{F}(t)$, and we shall do it here for $\mathbf{Q}$.

To show that $\mathbf{Q}^+$ is discrete in $\mathbb{A}_{\mathbf{Q}}^+$ it is enough, because of the group structure, to find an open set U that contains $0 \in \mathbb{A}_{\mathbf{Q}}^+$, but which contains no other elements of $\mathbf{Q}^+$. (If $\alpha \in \mathbf{Q}^+$, then $U + \alpha$ is an open subset of $\mathbb{A}_{\mathbf{Q}}^+$ whose intersection with $\mathbf{Q}^+$ is $\{\alpha\}$.) We take for U the set of $\mathbf{x} = \{x_v\} \in \mathbb{A}_{\mathbf{Q}}^+$ with

$$|x_\infty|_\infty < 1 \qquad \text{and} \qquad |x_p|_p \leq 1 \quad (\text{all } p),$$

where $|\cdot|_p$ and $|\cdot|_\infty$ are respectively the p-adic and the usual archimedean absolute values on $\mathbf{Q}$. If $b \in \mathbf{Q} \cap U$, then in the first place $b \in \mathbf{Z}$ because $|b|_p \leq$ for all p, and then $b = 0$ because $|b|_\infty < 1$. This proves that K^+ is discrete in $\mathbb{A}_{\mathbf{Q}}^+$.

Next we prove that the quotient $\mathbb{A}_{\mathbf{Q}}^+/\mathbf{Q}^+$ is compact. Let $W \subset \mathbb{A}_{\mathbf{Q}}^+$ consist of the $\mathbf{x} = \{x_v\} \in \mathbb{A}_{\mathbf{Q}}^+$ with

$$|x_\infty|_\infty \leq \frac{1}{2} \qquad \text{and} \qquad |x_p|_p \leq 1 \qquad \text{for all primes } p.$$

We show that every adele $\mathbf{y} = \{y_v\}$ is of the form

$$\mathbf{y} = a + \mathbf{x}, \qquad a \in \mathbf{Q}, \quad \mathbf{x} \in W.$$

Fix an adele $\mathbf{y} = \{y_v\} \in \mathbb{A}_{\mathbf{Q}}^{+}$. For each prime p we can find a rational number

$$r_p = \frac{z_p}{p^{n_p}} \qquad (z_p \in \mathbf{Z}, \quad n_p \in \mathbf{Z}_{\geq 0})$$

such that

$$|y_p - r_p|_p \leq 1,$$

and

$$r_p \equiv 0 \pmod{p} \qquad \text{for almost all } p,$$

since $\mathbf{y}$ is an adele. More precisely, for the finitely many p such that

$$y_p = \sum_{n \geq -|s|} a_n p^n \notin \mathbf{Z}_p,$$

choose r_p to be a rational number that is the value of an appropriate truncation of the p-adic expansion of y_p, and when $y_p \in \mathbf{Z}_p$ just choose $r_p = 0$. Hence $r = \sum_p r_p \in \mathbf{Q}$ is well defined and

$$|y_p - r|_p \leq 1 \qquad \text{for all } p.$$

(The r_q for $q \neq p$ do not mess up the inequality $|y_p - r|_p \leq 1$ since the valuation is non-archimedean and the r_q do not have any p in their denominator, by construction.) Now choose $s \in \mathbf{Z}$ such that

$$|b_\infty - r - s| \leq \frac{1}{2}.$$

Then $a = r + s$ and $\mathbf{x} = \mathbf{y} - a$ do what is required, since $\mathbf{y} - a = \mathbf{y} - r - s$ has the desired property (since $s \in \mathbf{Z}$ and the p-adic valuations are non-archimedean).

Hence the continuous map $W \to \mathbb{A}_{\mathbf{Q}}^{+}/\mathbf{Q}^{+}$ induced by the quotient map $\mathbb{A}_{\mathbf{Q}}^{+} \to \mathbb{A}_{\mathbf{Q}}^{+}/\mathbf{Q}^{+}$ is surjective. But W is compact (being the topological product of the compact spaces $|x_\infty|_\infty \leq 1/2$ and the $\mathbf{Z}_p$ for all p), hence $\mathbb{A}_{\mathbf{Q}}^{+}/\mathbf{Q}^{+}$ is also compact. $\square$

As already remarked, $\mathbb{A}_K^{+}$ is a locally compact group, so it has an invariant Haar measure. In fact one choice of this Haar measure is the product of the Haar measures on the K_v, in the sense described in the previous section.

Corollary 14.6. *There is a subset W of $\mathbb{A}_K$ defined by inequalities of the type $|x_v|_v \leq \delta_v$, where $\delta_v = 1$ for almost all v, such that every $\mathbf{y} \in \mathbb{A}_K$ can be put in the form*

$$\mathbf{y} = a + \mathbf{x}, \qquad a \in K, \quad \mathbf{x} \in W,$$

i.e., $\mathbb{A}_K = K + W$.

Proof. We constructed such a set for $K = \mathbf{Q}$ when proving Theorem 14.5. For general K the W coming from the proof determines compenent-wise a subset of $\mathbb{A}_K^{+} \cong \mathbb{A}_{\mathbf{Q}}^{+} \oplus \cdots \oplus \mathbb{A}_{\mathbf{Q}}^{+}$ that is a subset of a W with the properties claimed by the corollary. $\square$

Corollary 14.7. *The quotient $\mathbb{A}_K^+/K^+$ has finite measure in the quotient measure induced by the Haar measure on $\mathbb{A}_K^+$.*

Remark 14.8. This statement is independent of the particular choice of the multiplicative constant in the Haar measure on $\mathbb{A}_K^+$. We do not here go into the question of finding the measure $\mathbb{A}_K^+/K^+$ in terms of our explicitly given Haar measure. (See Tate's thesis, Chapter XV of Cassels-Frohlich.)

Proof. This can be reduced similarly to the case of $\mathbf{Q}$ or $\mathbf{F}(t)$ which is immediate, e.g., the W defined above has measure 1 for our Haar measure.

Alternatively, finite measure follows from compactness. To see this, cover $\mathbb{A}_K/K^+$ with the translates of U, where U is a nonempty open set with finite measure. The existence of a finite subcover implies finite measure. $\square$

Remark 14.9. We give an alternative proof of the product formula $\prod |a|_v = 1$ for nonzero $a \in K$. We have seen that if $x_v \in K_v$, then multiplication by x_v magnifies the Haar measure in K_v^+ by a factor of $|x_v|_v$. Hence if $\mathbf{x} = \{x_v\} \in \mathbb{A}_K$, then multiplication by $\mathbf{x}$ magnifies the Haar measure in $\mathbb{A}_K^+$ by $\prod |x_v|_v$. But now multiplication by $a \in K$ takes $K^+ \subset \mathbb{A}_K^+$ into K^+, so gives a well-defined bijection of $\mathbb{A}_K^+/K^+$ onto $\mathbb{A}_K^+/K^+$ which magnifies the measure by the factor $\prod |a|_v$. Hence $\prod |a|_v = 1$ Corollary 14.7. (The point is that if μ is the measure of $\mathbb{A}_K^+/K^+$, then $\mu = \prod |a|_v \cdot \mu$, so because μ is finite we must have $\prod |a|_v = 1$.)

Lemma 14.10. *There is a constant $C > 0$ depending only on the global field K with the following property:*

Whenever $\mathbf{x} = \{x_v\}_v \in \mathbb{A}_K$ is such that

$$\prod_v |x_v|_v > C,$$

then there is a nonzero principal adele $a \in K \subset \mathbb{A}_K$ such that

$$|a|_v \leq |x_v|_v \qquad \textit{(all } v\textit{).}$$

Proof. $\square$

Corollary 14.11. *Let v_0 be a normalized valuation and let $\delta_v > 0$ be given for all $v \neq v_0$ with $\delta_v = 1$ for almost all v. Then there is a nonzero $a \in K$ with*

$$|a|_v \leq \delta_v \qquad \textit{(all } v \neq v_0\textit{).}$$

Proof. $\square$

Next week we will prove the above two lemmas, then use them to deduce the strong approximation theorem, which is an extreme generalization of the Chinese Remainder Theorem; it asserts that K^+ is *dense* (!) in the analogue of the adeles but with one places removed. Then we'll introduce the ideles $\mathbb{A}_K^*$. Finally on Thursday, we'll relate ideles to ideals, and use everything so far to give a new interpretation of class groups and their finiteness.