Math 129: Algebraic Number Theory
Lecture 21: Global Fields and Adeles

William Stein (based closely on Cassels's *Global Fields* article in Cassels-Fröhlich)

Tuesday, April 27, 2004

12 Global Fields

Definition 12.1 (Global Field). A *global field* is a number field or a finite separable extension of $\mathbf{F}(t)$, where $\mathbf{F}$ is a finite field, and t is transcendental over $\mathbf{F}$.

Below we will focus attention on number fields leaving the function field case to the reader.

The following lemma essentially says that the denominator of an element of a global field is only "nontrivial" at a finite number of valuations.

Lemma 12.2. *Let $a \in K$ be a nonzero element of a global field K. Then there are only finitely many inequivalent valuations $|\cdot|$ of K for which*

$$|a| > 1.$$

Proof. If $K = \mathbf{Q}$ or $\mathbf{F}(t)$ then the lemma follows by Ostrowski's classification of all the valuations on K. For example, when $a = \frac{n}{d} \in \mathbf{Q}$, with $n, d \in \mathbf{Z}$, then the valuations where we could have $|a| > 1$ are the archimedean one, or the p-adic valuations $|\cdot|_p$ for which $p \mid d$.

Suppose now that K is a finite extension of $\mathbf{Q}$, so a satisfies a monic polynomial

$$a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0,$$

for some n and $c_0, \ldots, c_{n-1} \in \mathbf{Q}$. If $|\cdot|$ is a non-archimedean valuation on K, we have

$$\begin{aligned} |a|^n &= \left|-(c_{n-1}a^{n-1} + \cdots + c_0)\right| \\ &\leq \max(1, |a|^{n-1}) \cdot \max(|c_0|, \ldots, |c_{n-1}|). \end{aligned}$$

Dividing each side by $|a|^{n-1}$, we have that

$$|a| \leq \max(|c_0|, \ldots, |c_{n-1}|),$$

so in all cases we have

$$|a| \leq \max(1, |c_0|, \ldots, |c_{n-1}|)^{1/(n-1)}. \tag{12.1}$$

We know the lemma for $\mathbf{Q}$, so there are only finitely many valuations $|\cdot|$ on $\mathbf{Q}$ such that the right hand side of (12.1) is bigger than 1. Since each valuation of $\mathbf{Q}$ has finitely many extensions to K, and there are only finitely many archimedean valuations, it follows that there are only finitely many valuations on K such that $|a| > 1$. □

Any valuation on a global field is either archimedean, or discrete non-archimedean with finite residue class field, since this is true of $\mathbf{Q}$ and $\mathbf{F}(t)$ and is a property preserved by extending a valuation to a finite extension of the base field. Hence it makes sense to talk of normalized valuations. Recall that the normalized p-adic valuation on $\mathbf{Q}$ is $|x|_p = p^{-\operatorname{ord}_p(x)}$, and if v is a valuation on a number field K equivalent to an extension of $|\cdot|_p$, then the normalization of v is the composite of the sequence of maps

$$K \hookrightarrow K_v \xrightarrow{\text{Norm}} \mathbf{Q}_p \xrightarrow{|\cdot|_p} \mathbf{R},$$

where K_v is the completion of K at v.

Example 12.3. Let $K = \mathbf{Q}(\sqrt{2})$, and let $p = 2$. Because $\sqrt{2} \notin \mathbf{Q}_2$, there is exactly one extension of $|\cdot|_2$ to K, and it sends $a = 1/\sqrt{2}$ to

$$\left|\operatorname{Norm}_{\mathbf{Q}_2(\sqrt{2})/\mathbf{Q}_2}(1/\sqrt{2})\right|_2^{1/2} = \sqrt{2}.$$

Thus the normalized valuation of a is 2.

There are two extensions of $|\cdot|_7$ to $\mathbf{Q}(\sqrt{2})$, since $\mathbf{Q}(\sqrt{2}) \otimes_{\mathbf{Q}} \mathbf{Q}_7 \cong \mathbf{Q}_7 \oplus \mathbf{Q}_7$, as $x^2 - 2 = (x-3)(x-4) \pmod 7$. The image of $\sqrt{2}$ under each embedding into $\mathbf{Q}_7$ is a unit in $\mathbf{Z}_7$, so the normalized valuation of $a = 1/\sqrt{2}$ is, in both cases, equal to 1. More generally, for any valuation of K of characteristic an odd prime p, the normalized valuation of a is 1.

Since $K = \mathbf{Q}(\sqrt{2}) \hookrightarrow \mathbf{R}$ in two ways, there are exactly two normalized archimedean valuations on K, and both of their values on a equal $1/\sqrt{2}$. Notice that the product of the absolute values of a with respect to all normalized valuations is

$$2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot 1 \cdot 1 \cdot 1 \cdots = 1.$$

This "product formula" holds in much more generality, as we will now see.

Theorem 12.4 (Product Formula). *Let $a \in K$ be a nonzero element of a global field K. Let $|\cdot|_v$ run through the normalized valuations of K. Then $|a|_v = 1$ for almost all v, and*

$$\prod_{\text{all } v} |a|_v = 1 \qquad \text{(the product formula)}.$$

Proof. By Lemma 12.2, we have $|a|_v \leq 1$ for almost all v. Likewise, $1/|a|_v = |1/a|_v \leq 1$ for almost all v, so $|a|_v = 1$ for almost all v.

Let w run through all normalized valuations of $\mathbf{Q}$ (or of $\mathbf{F}(t)$), and write $v \mid w$ if the restriction of v to $\mathbf{Q}$ is equivalent to w. Then (by the previous section),

$$\prod_v |a|_v = \prod_w \left(\prod_{v|w} |a|_v\right) = \prod_w \left|\operatorname{Norm}_{K/\mathbf{Q}}(a)\right|_w,$$

so it suffices to prove the theorem for $K = \mathbf{Q}$.

By multiplicativity of valuations, if the theorem is true for b and c then it is true for the product bc and quotient b/c (when $c \neq 0$). The theorem is clearly true for -1, which has valuation 1 at all valuations. Thus to prove the theorem for $\mathbf{Q}$ it suffices to prove it when $a = p$ is a prime number. Then we have $|p|_\infty = p$, $|p|_p = 1/p$, and for primes $q \neq p$ that $|p|_q = 1$. Thus

$$\prod_v |p|_v = p \cdot \frac{1}{p} \cdot 1 \cdot 1 \cdot 1 \cdots = 1,$$

as claimed. □

If v is a valuation on a field K, recall that we let K_v denote the completion of K with respect to v. Also when v is non-archimedean, let

$$\mathcal{O}_v = \mathcal{O}_{K,v} = \{x \in K_v : |x| \leq 1\}$$

be the ring of integers of the completion.

Definition 12.5 (Almost All). We say a condition holds for *almost all* elements of a set if it holds for all but finitely many elements.

We will use the following lemma later to prove that formation of the adeles of a global field is compatible with base change.

Lemma 12.6. *Let $\omega_1, \ldots, \omega_n$ be a basis for L/K, where L is a finite separable extension of the global field K of degree n. Then for almost all normalized non-archimedean valuations v on K we have*

$$\omega_1 \mathcal{O}_v \oplus \cdots \oplus \omega_n \mathcal{O}_v = \mathcal{O}_{w_1} \oplus \cdots \oplus \mathcal{O}_{w_g} \subset K_v \otimes_K L, \tag{12.2}$$

where $w_1, \ldots, w_g$ are the extensions of v to L. Here we have identified $a \in L$ with its canonical image in $K_v \otimes_K L$, and the direct sum on the left is the sum taken inside the tensor product (so directness means that the intersections are trivial).

Proof. The proof proceeds in two steps. First we deduce easily from Lemma 12.2 that for almost all v the left hand side of (12.2) is contained in the right hand side. Then we use a trick involving discriminants to show the opposite inclusion for all but finitely many primes.

Since $\mathcal{O}_v \subset \mathcal{O}_{w_i}$ for all i, the left hand side of (12.2) is contained in the right hand side if $|\omega_i|_{w_j} \leq 1$ for $1 \leq i \leq n$ and $1 \leq j \leq g$. Thus by Lemma 12.2, for all but finitely many v the left hand side of (12.2) is contained in the right hand side. We have just eliminated the finitely many primes corresponding to "denominators" of some ω_i, and now only consider v such that $\omega_1, \ldots, \omega_n \in \mathcal{O}_w$ for all $w \mid v$.

For any elements $a_1, \ldots, a_n \in K_v \otimes_K L$, consider the discriminant

$$D(a_1, \ldots, a_n) = \mathrm{Det}(\mathrm{Tr}(a_i a_j)) \in K_v,$$

where the trace is induced from the L/K trace. Since each ω_i is in each $\mathcal{O}_w$, for $w \mid v$, the traces lie in $\mathcal{O}_v$, so

$$d = D(\omega_1, \ldots, \omega_n) \in \mathcal{O}_v.$$

Also note that $d \in K$ since each ω_i is in L. Now suppose that

$$\alpha = \sum_{i=1}^{n} a_i \omega_i \in \mathcal{O}_{w_1} \oplus \cdots \oplus \mathcal{O}_{w_g},$$

with $a_i \in K_v$. Then by properties of determinants for any m with $1 \leq m \leq n$, we have

$$D(\omega_1, \ldots, \omega_{m-1}, \alpha, \omega_{m+1}, \ldots, \omega_n) = a_m^2 D(\omega_1, \ldots, \omega_n). \tag{12.3}$$

The left hand side of (12.3) is in $\mathcal{O}_v$, so the right hand side is well, i.e.,

$$a_m^2 \cdot d \in \mathcal{O}_v, \qquad \text{(for } m = 1, \ldots, n),$$

where $d \in K$. Since $\omega_1, \ldots, \omega_n$ are a basis for L over K and the trace pairing is nondegenerate, we have $d \neq 0$, so by Theorem 12.4 we have $|d|_v = 1$ for all but finitely many v. Then for all but finitely many v we have that $a_m^2 \in \mathcal{O}_v$. For these v, that $a_m^2 \in \mathcal{O}_v$ implies $a_m \in \mathcal{O}_v$ since $a_m \in K_v$, i.e., α is in the left hand side of (12.2). □

Example 12.7. Let $K = \mathbf{Q}$ and $L = \mathbf{Q}(\sqrt{2})$. Let $\omega_1 = 1/3$ and $\omega_2 = 2\sqrt{2}$. In the first stage of the above proof we would eliminate $|\cdot|_3$ because ω_2 is not integral at 3. The discriminant is

$$d = D\left(\frac{1}{3}, 2\sqrt{2}\right) = \mathrm{Det}\begin{pmatrix} \frac{2}{9} & 0 \\ 0 & 16 \end{pmatrix} = \frac{32}{9}.$$

As explained in the second part of the proof, as long as $v \neq 2, 3$, we have equality of the left and right hand sides in (12.2).

13 Restricted Topological Products

In this section we describe a topological tool, which we need in order to define adeles (see Definition 14.1).

Definition 13.1 (Restricted Topological Products). Let X_λ, for $\lambda \in \Lambda$, be a family of topological spaces, and for almost all λ let $Y_\lambda \subset X_\lambda$ be an open subset of X_λ. Consider the space X whose elements are sequences $\mathbf{x} = \{x_\lambda\}_{\lambda \in \Lambda}$, where $x_\lambda \in X_\lambda$ for every λ, and $x_\lambda \in Y_\lambda$ for almost all λ. We give X a topology by taking as a basis of open sets the sets $\prod U_\lambda$, where $U_\lambda \subset X_\lambda$ is open for all λ, and $U_\lambda = Y_\lambda$ for almost all λ. We call X with this topology the *restricted topological product* of the X_λ with respect to the Y_λ.

Corollary 13.2. *Let S be a finite subset of Λ, and let X_S be the set of $\mathbf{x} \in X$ with $x_\lambda \in Y_\lambda$ for all $\lambda \notin S$, i.e.,*

$$X_S = \prod_{\lambda \in S} X_\lambda \times \prod_{\lambda \notin S} Y_\lambda \subset X.$$

Then X_S is an open subset of X, and the topology induced on X_S as a subset of X is the same as the product topology.

The restricted topological product depends on the totality of the Y_λ, but not on the individual Y_λ:

Lemma 13.3. *Let $Y'_\lambda \subset X_\lambda$ be open subsets, and suppose that $Y_\lambda = Y'_\lambda$ for almost all λ. Then the restricted topological product of the X_λ with respect to the Y'_λ is canonically isomorphic to the restricted topological product with respect to the Y_λ.*

Lemma 13.4. *Suppose that the X_λ are locally compact and that the Y_λ are compact. Then the restricted topological product X of the X_λ is locally compact.*

Proof. For any finite subset S of Λ, the open subset $X_S \subset X$ is locally compact, because by Lemma 13.2 it is a product of finitely many locally compact sets with an infinite product of compact sets. (Here we are using Tychonoff's theorem from topology, which asserts that an arbitrary product of compact topological spaces is compact (see Munkres's *Topology, a first course*, chapter 5).) Since $X = \cup_S X_S$, and the X_S are open in X, the result follows. □

Definition 13.5 (Product Measure). For all $\lambda \in \Lambda$, suppose μ_λ is a measure on X_λ with $\mu_\lambda(Y_\lambda) = 1$ when Y_λ is defined. We define the *product measure* μ on X to be that for which a basis of measurable sets is

$$\prod_\lambda M_\lambda$$

where each $M_\lambda \subset X_\lambda$ has finite μ_λ-measure and $M_\lambda = Y_\lambda$ for almost all λ, and where

$$\mu\left(\prod_\lambda M_\lambda\right) = \prod_\lambda \mu_\lambda(M_\lambda).$$

14 The Adele Ring

Let K be a global field. For each normalization $|\cdot|_v$ of K, let K_v denote the completion of K. If $|\cdot|_v$ is non-archimedean, let $\mathcal{O}_v$ denote the ring of integers of K_v.

Definition 14.1 (Adele Ring). The *adele ring* $\mathbb{A}_K$ of K is the topological ring whose underlying topological space is the restricted topological product of the K_v with respect to the $\mathcal{O}_v$, and where addition and multiplication are defined componentwise:

$$(\mathbf{x}\mathbf{y})_v = \mathbf{x}_v\mathbf{y}_v \qquad (\mathbf{x}+\mathbf{y})_v = \mathbf{x}_v + \mathbf{y}_v \qquad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{A}_K. \tag{14.1}$$

It is readily verified that (i) this definition makes sense, i.e., if $\mathbf{x}, \mathbf{y} \in \mathbb{A}_K$, then $\mathbf{x}\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$, whose components are given by (14.1), are also in $\mathbb{A}_K$, and (ii) that addition and multiplication are continuous in the $\mathbb{A}_K$-topology, so $\mathbb{A}_K$ is a topological ring, as asserted. Also, $\mathbb{A}_K$ is locally compact because the K_v are locally compact, and the $\mathcal{O}_v$ are compact.

There is a natural continuous ring inclusion

$$K \hookrightarrow \mathbb{A}_K \tag{14.2}$$

that sends $x \in K$ to the adele every one of whose components is x. This is an adele because $x \in \mathcal{O}_v$ for almost all v, by Lemma 12.2. The map is injective because each map $K \to K_v$ is an inclusion.

Definition 14.2 (Principal Adeles). The image of (14.2) is the ring of *principal adeles*.

It will cause no trouble to identify K with the principal adeles, so we shall speak of K as a subring of $\mathbb{A}_K$.

Formation of the adeles is compatibility with base change, in the following sense.

Lemma 14.3. *Suppose L is a finite (separable) extension of the global field K. Then*

$$\mathbb{A}_K \otimes_K L \cong \mathbb{A}_L$$

both algebraically and topologically. Under this isomorphism, $L \cong K \otimes_K L \subset \mathbb{A}_K \otimes_K L$ maps isomorphically onto $L \subset \mathbb{A}_L$.

* * *

Next time we will start by proving the above lemma. Here are some further highlights of what will come in the final three lectures.

Theorem 14.4. *The global field K is discrete in $\mathbb{A}_K$ and the quotient $\mathbb{A}_K^+/K^+$ of additive groups is compact in the quotient topology.*

If we remove even one valuation, then the situation changes dramatically:

Theorem 14.5 (Strong Approximation). *Let v_0 be any valuation of the global field K. Define $\mathbb{A}'_K$ to be the restricted topological product of the K_v with respect to the $\mathcal{O}_v$, where v runs through all normalized valuations $v \neq v_0$. Then K is dense in $\mathbb{A}'_K$.*

Definition 14.6 (Idele Group). The idele group $\mathbb{I}_K$ of K is the group $\mathbb{A}_K^*$ of invertible elements of the adele ring $\mathbb{A}_K$.

The subgroup $\mathbb{I}_K^1$ is the subgroup of ideles $\mathbf{x} = \{x_v\}$ such that $\prod_v |x_v|_v = 1$. Note that $K^* \subset \mathbb{I}_K^1$.

Theorem 14.7. *The quotient $\mathbb{I}_K^1/K^*$ with the quotient topology is compact.*

Theorem 14.8. *The ideal class group of K (with the discrete topology) is canonically isomorphic to $\mathbb{I}_K^1/K^*$.*

Since a discrete compact group is finite, this proves that the ideal class group of a global field is finite.

I think we will likely stop here, and not do sections 18 and 19. This will be a nice conclusion, because you'll finish the class having learned the basic theorems and objects of algebraic number theory, both from the classical and adelic points of view, and will have seen a nontrivial result proved from both directions. You have also learned something about the structure of local fields (completions of global fields).