Math 129: Algebraic Number Theory
Homework Assignment 10

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Due: Thursday, April 29, 2004

1. Prove that the ring C defined in Section 9 really is the tensor product of A and B, i.e., that it satisfies the defining universal mapping property for tensor products. Part of this problem is for you to look up a functorial definition of tensor product.

2. Find a zero divisor pair in $\mathbf{Q}(\sqrt{5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{5})$.

3. (a) Is $\mathbf{Q}(\sqrt{5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{-5})$ a field?
   (b) Is $\mathbf{Q}(\sqrt[4]{5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt[4]{-5}) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{-1})$ a field?

4. Suppose ζ_5 denotes a primitive 5th root of unity. For any prime p, consider the tensor product $\mathbf{Q}_p \otimes_{\mathbf{Q}} \mathbf{Q}(\zeta_5) = K_1 \oplus \cdots \oplus K_{n(p)}$. Find a simple formula for the number $n(p)$ of fields appearing in the decomposition of the tensor product $\mathbf{Q}_p \otimes_{\mathbf{Q}} \mathbf{Q}(\zeta_5)$. To get full credit on this problem your formula must be correct, but you do *not* have to prove that it is correct.

5. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a finite-dimensional vector space V over a field K (with valuation $|\cdot|$). Carefully prove that the topology induced by $\|\cdot\|_1$ is the same as that induced by $\|\cdot\|_2$.

6. Suppose K and L are number fields (i.e., finite extensions of $\mathbf{Q}$). Is it possible for the tensor product $K \otimes_{\mathbf{Q}} L$ to contain a nilpotent element? (A nonzero element a in a ring R is *nilpotent* if there exists $n > 1$ such that $a^n = 0$.)

7. Let K be the number field $\mathbf{Q}(\sqrt[5]{2})$.

   (a) In how many ways does the 2-adic valuation $|\cdot|_2$ on $\mathbf{Q}$ extend to a valuation on K?

   (b) Let $v = |\cdot|$ be a valuation on K that extends $|\cdot|_2$. Let K_v be the completion of K with respect to v. What is the residue class field $\mathbf{F}$ of K_v?