2.5 Visibility of Shafarevich-Tate Groups

Let K be a number field. Suppose

$$0 \to A \to B \to C \to 0$$

is an exact sequence of abelian varieties over K. (Thus each of A, B, and C is a complete group variety over K, whose group is automatically abelian.) Then there is a corresponding long exact sequence of cohomology for the group $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$:

$$0 \to A(K) \to B(K) \to C(K) \xrightarrow{\delta} \mathrm{H}^1(K, A) \to \mathrm{H}^1(K, B) \to \mathrm{H}^1(K, C) \to \cdots$$

The study of the Mordell-Weil group $C(K) = \mathrm{H}^0(K, C)$ is popular in arithmetic geometry. For example, the Birch and Swinnerton-Dyer conjecture (BSD conjecture), which is one of the million dollar Clay Math Problems, asserts that the dimension of $C(K) \otimes \mathbf{Q}$ equals the ordering vanishing of $L(C, s)$ at $s = 1$.

The group $\mathrm{H}^1(K, A)$ is also of interest in connection with the BSD conjecture, because it contains the Shafarevich-Tate group

$$Ш(A) = Ш(A/K) = \mathrm{Ker}\left(\mathrm{H}^1(K, A) \to \bigoplus_v \mathrm{H}^1(K_v, A)\right) \subset \mathrm{H}^1(K, A),$$

where the sum is over all places v of K (e.g., when $K = \mathbf{Q}$, the fields K_v are $\mathbf{Q}_p$ for all prime numbers p and $\mathbf{Q}_\infty = \mathbf{R}$).

The group $A(K)$ is fundamentally different than $\mathrm{H}^1(K, C)$. The Mordell-Weil group $A(K)$ is finitely generated, whereas the first Galois cohomology $\mathrm{H}^1(K, C)$ is far from being finitely generated—in fact, every element has finite order and there are infinitely many elements of any given order.

This talk is about "dimension shifting", i.e., relating information about $\mathrm{H}^0(K, C)$ to information about $\mathrm{H}^1(K, A)$.

2.5.1 Definitions

Elements of $\mathrm{H}^0(K, C)$ are simply points, i.e., elements of $C(K)$, so they are relatively easy to "visualize". In contrast, elements of $\mathrm{H}^1(K, A)$ are Galois cohomology classes, i.e., equivalence classes of set-theoretic (continuous) maps $f : \mathrm{Gal}(\overline{\mathbf{Q}}/K) \to A(\overline{\mathbf{Q}})$ such that $f(\sigma\tau) = f(\sigma) + \sigma f(\tau)$. Two maps are equivalent if their difference is a map of the form $\sigma \mapsto \sigma(P) - P$ for some fixed $P \in A(\overline{\mathbf{Q}})$. From this point of view H^1 is more mysterious than H^0.

There is an alternative way to view elements of $\mathrm{H}^1(K, A)$. The WC group of A is the group of isomorphism classes of principal homogeneous spaces for A, where a principal homogeneous space is a variety X and a map $A \times X \to X$ that satisfies the same axioms as those for a simply transitive group action. Thus X is a twist as variety of A, but $X(K) = \emptyset$, unless $X \approx A$. Also, the nontrivial elements of $Ш(A)$ correspond to the classes in WC that have a K_v-rational point for all places v, but no K-rational point.

Mazur introduced the following definition in order to help unify diverse constructions of principal homogeneous spaces:

Definition 2.5.1 (Visible). The *visible subgroup* of $\mathrm{H}^1(K, A)$ in B is

$$\begin{aligned} \mathrm{Vis}_B \mathrm{H}^1(K, A) &= \mathrm{Ker}(\mathrm{H}^1(K, A) \to \mathrm{H}^1(K, B)) \\ &= \mathrm{Coker}(B(K) \to C(K)). \end{aligned}$$

Remark 2.5.2. Note that $\mathrm{Vis}_B \mathrm{H}^1(K, A)$ *does* depend on the embedding of A into B. For example, suppose $B = B_1 \times A$. Then there could be nonzero visible elements if A is embedding into the first factor, but there will be no nonzero visible elements if A is embedded into the second factor. Here we are using that $\mathrm{H}^1(K, B_1 \times A) = \mathrm{H}^1(K, B_1) \oplus \mathrm{H}^1(K, A)$.

The connection with the WC group of A is as follows. Suppose

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence of abelian varieties and that $c \in \mathrm{H}^1(K, A)$ is visible in B. Thus there exists $x \in C(K)$ such that $\delta(x) = c$, where $\delta : C(K) \to \mathrm{H}^1(K, A)$ is the connecting homomorphism. Then $X = \pi^{-1}(x) \subset B$ is a translate of A in B, so the group law on B gives X the structure of principal homogeneous space for A, and one can show that the class of X in the WC group of A corresponds to c.

Lemma 2.5.3. *The group* $\mathrm{Vis}_B \mathrm{H}^1(K, A)$ *is finite.*

Proof. Since $\mathrm{Vis}_B \mathrm{H}^1(K, A)$ is a homomorphic image of the finitely generated group $C(K)$, it is also finitely generated. On the other hand, it is a subgroup of $\mathrm{H}^1(K, A)$, so it is a torsion group. The lemma follows since a finitely generated torsion abelian group is finite. □

2.5.2 Every Element of $\mathrm{H}^1(K, A)$ *is Visible Somewhere*

Proposition 2.5.4. *Let* $c \in \mathrm{H}^1(K, A)$*. Then there exists an abelian variety* $B = B_c$ *and an embedding* $A \hookrightarrow B$ *such that* c *is visible in* B*.*

Proof. By definition of Galois cohomology, there is a finite extension L of K such that $\mathrm{res}_L(c) = 0$. Thus c maps to 0 in $\mathrm{H}^1(L, A_L)$. By a slight generalization of the Shapiro Lemma from group cohomology (which can be proved by dimension shifting; see, e.g., Atiyah-Wall in Cassels-Frohlich), there is a canonical isomorphism

$$\mathrm{H}^1(L, A_L) \cong \mathrm{H}^1(K, \mathrm{Res}_{L/K}(A_L)) = \mathrm{H}^1(K, B),$$

where $B = \mathrm{Res}_{L/K}(A_L)$ is the Weil restriction of scalars of A_L back down to K. The restriction of scalars B is an abelian variety of dimension $[L : K] \cdot \dim A$ that is characterized by the existence of functorial isomorphisms

$$\mathrm{Mor}_K(S, B) \cong \mathrm{Mor}_L(S_L, A_L),$$

for any K-scheme S, i.e., $B(S) = A_L(S_L)$. In particular, setting $S = A$ we find that the identity map $A_L \to A_L$ corresponds to an injection $A \hookrightarrow B$. Moreover, $c \mapsto \mathrm{res}_L(c) = 0 \in \mathrm{H}^1(K, B)$. □

Remark 2.5.5. The abelian variety B in Proposition 2.5.4 is a twist of a power of A.

2.5.3 Visibility in the Context of Modularity

Usually we focus on visibility of elements in $Ш(A)$. There are a number of other results about visibility in various special cases, and large tables of examples in the context of elliptic curves and modular abelian varieties. There are also interesting modularity questions and conjectures in this context.

Motivated by the desire to understand the Birch and Swinnerton-Dyer conjecture more explicitly, I developed (with significant input from Agashe, Cremona, Mazur, and Merel) computational techniques for unconditionally constructing Shafarevich-Tate groups of modular abelian varieties $A \subset J_0(N)$ (or $J_1(N)$). For example, if $A \subset J_0(389)$ is the 20-dimensional simple factor, then

$$\mathbf{Z}/5\mathbf{Z} \times \mathbf{Z}/5\mathbf{Z} \subset Ш(A),$$

as predicted by the Birch and Swinnerton-Dyer conjecture. See [CM00] for examples when $\dim A = 1$. We will spend the rest of this section discussing the examples of [ASb, AS02] in more detail.

Tables 2.5.1–2.5.4 illustrate the main computational results of [ASb]. These tables were made by gathering data about certain arithmetic invariants of the 19608 abelian varieties A_f of level ≤ 2333. Of these, exactly 10360 have satisfy $L(A_f, 1) \neq 0$, and for these with $L(A_f, 1) \neq 0$, we compute a divisor and multiple of the conjectural order of $Ш(A_f)$. We find that there are at least 168 such that the Birch and Swinnerton-Dyer Conjecture implies that $Ш(A_f)$ is divisible by an odd prime, and we prove for 37 of these that the odd part of the conjectural order of $Ш(A_f)$ really divides $\#Ш(A_f)$ by constructing nontrivial elements of $Ш(A_f)$ using visibility.

The meaning of the tables is as follows. The first column lists a level N and an isogeny class, which uniquely specifies an abelian variety $A = A_f \subset J_0(N)$. The nth isogeny class is given by the nth letter of the alphabet. We will not discuss the ordering further, except to note that usually, the dimension of A, which is given in the second column, is enough to determine A. When $L(A, 1) \neq 0$, Conjecture 2.2.1 predicts that

$$\#Ш(A) \stackrel{?}{=} \frac{L(A,1)}{\Omega_A} \cdot \frac{\#A(\mathbf{Q})_{\mathrm{tor}} \cdot \#A^{\vee}(\mathbf{Q})_{\mathrm{tor}}}{\prod_{p \mid N} c_p}.$$

We view the quotient $L(A, 1)/\Omega_A$, which is a rational number, as a single quantity. We can compute multiples and divisors of every quantity appearing in the right hand side of this equation, and this yields columns three and four, which are a divisor S_ℓ and a multiple S_u of the conjectural order of $Ш(A)$ (when $S_u = S_\ell$, we put an equals sign in the S_u column). Column five, which is labeled $\operatorname{odd}\deg(\varphi_A)$, contains the odd part of the degree of the polarization

$$\varphi_A : (A \hookrightarrow J_0(N) \cong J_0(N)^{\vee} \to A^{\vee}). \tag{2.5.1}$$

The second set of columns, columns six and seven, contain an abelian variety $B = B_g \subset J_0(N)$ such that $\#(A \cap B)$ is divisible by an odd prime divisor of S_ℓ and $L(B, 1) = 0$. When $\dim(B) = 1$, we have verified that B is an elliptic curve of rank 2. The eighth column $A \cap B$ contains the group structure of $A \cap B$, where e.g., $[2^2 302^2]$ is shorthand notation for $(\mathbf{Z}/2\mathbf{Z})^2 \oplus (\mathbf{Z}/302\mathbf{Z})^2$. The final column, labeled Vis, contains a divisor of the order of $\mathrm{Vis}_{A+B}(Ш(A))$.

The following proposition explains the significance of the $\operatorname{odd}\deg(\varphi_A)$ column.

Proposition 2.5.6. *If $p \nmid \deg(\varphi_A)$, then $p \nmid \mathrm{Vis}_{J_0(N)}(\mathrm{H}^1(\mathbf{Q}, A))$.*

Proof. There exists a complementary morphism $\hat{\varphi}_A$, such that $\varphi_A \circ \hat{\varphi}_A = \hat{\varphi}_A \circ \varphi_A = [n]$, where n is the degree of φ_A. If $c \in \mathrm{H}^1(\mathbf{Q}, A)$ maps to 0 in $\mathrm{H}^1(\mathbf{Q}, J_0(N))$, then it also maps to 0 under the following composition

$$\mathrm{H}^1(\mathbf{Q}, A) \to \mathrm{H}^1(\mathbf{Q}, J_0(N)) \to \mathrm{H}^1(\mathbf{Q}, A^\vee) \xrightarrow{\hat{\varphi}_A} \mathrm{H}^1(\mathbf{Q}, A).$$

Since this composition is $[n]$, it follows that $c \in \mathrm{H}^1(\mathbf{Q}, A)[n]$, which proves the proposition. □

Remark 2.5.7. Since the degree of φ_A does not change if we extend scalars to a number field K, the subgroup of $\mathrm{H}^1(K, A)$ visible in $J_0(N)_K$, still has order divisible only by primes that divide $\deg(\varphi_A)$.

The following theorem explains the significance of the B column, and how it was used to deduce the Vis column.

Theorem 2.5.8. *Suppose A and B are abelian subvarieties of an abelian variety C over $\mathbf{Q}$ and that $A(\overline{\mathbf{Q}}) \cap B(\overline{\mathbf{Q}})$ is finite. Assume also that $A(\mathbf{Q})$ is finite. Let N be an integer divisible by the residue characteristics of primes of bad reduction for C (e.g., N could be the conductor of C). Suppose p is a prime such that*

$$p \nmid 2 \cdot N \cdot \#((A+B)/B)(\mathbf{Q})_{\mathrm{tor}} \cdot \#B(\mathbf{Q})_{\mathrm{tor}} \cdot \prod_{\ell} c_{A,\ell} \cdot c_{B,\ell},$$

where $c_{A,\ell} = \#\Phi_{A,\ell}(\mathbf{F}_\ell)$ is the Tamagawa number of A at ℓ (and similarly for B). Suppose furthermore that $B(\overline{\mathbf{Q}})[p] \subset A(\overline{\mathbf{Q}})$ as subgroups of $C(\overline{\mathbf{Q}})$. Then there is a natural injection

$$B(\mathbf{Q})/pB(\mathbf{Q}) \hookrightarrow \mathrm{Vis}_C(Ш(A)).$$

A complete proof of a generalization of this theorem can be found in [AS02].

Sketch of Proof. Without loss of generality, we may assume $C = A + B$. Our hypotheses yield a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B[p] & \longrightarrow & B & \xrightarrow{p} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B' & \longrightarrow & 0,
\end{array}$$

where $B' = C/A$. Taking $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-cohomology, we obtain the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B(\mathbf{Q}) & \xrightarrow{p} & B(\mathbf{Q}) & \longrightarrow & B(\mathbf{Q})/pB(\mathbf{Q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
0 & \longrightarrow & C(\mathbf{Q})/A(\mathbf{Q}) & \longrightarrow & B'(\mathbf{Q}) & \longrightarrow & \mathrm{Vis}_C(\mathrm{H}^1(\mathbf{Q}, A)) & \longrightarrow & 0.
\end{array}$$

The snake lemma and our hypothesis that $p \nmid \#(C/B)(\mathbf{Q})_{\mathrm{tor}}$ imply that the rightmost vertical map is an injection

$$i : B(\mathbf{Q})/pB(\mathbf{Q}) \hookrightarrow \mathrm{Vis}_C(\mathrm{H}^1(\mathbf{Q}, A)), \tag{2.5.2}$$

since $C(A)/(A(\mathbf{Q}) + B(\mathbf{Q}))$ is a sub-quotient of $(C'/B)(\mathbf{Q})$.

We show that the image of (2.5.2) lies in $Ш(A)$ using a local analysis at each prime, which we now sketch. At the archimedian prime, no work is needed since $p \neq 2$. At non-archimedian primes ℓ, one uses facts about Néron models (when $\ell = p$) and our hypothesis that p does not divide the Tamagawa numbers of B (when $\ell \neq p$) to show that if $x \in B(\mathbf{Q})/pB(\mathbf{Q})$, then the corresponding cohomology class $\mathrm{res}_\ell(i(x)) \in \mathrm{H}^1(\mathbf{Q}_\ell, A)$ splits over the maximal unramified extension. However,

$$\mathrm{H}^1(\mathbf{Q}_\ell^{\mathrm{ur}}/\mathbf{Q}_\ell, A) \cong \mathrm{H}^1(\overline{\mathbf{F}}_\ell/\mathbf{F}_\ell, \Phi_{A,\ell}(\overline{\mathbf{F}}_\ell)),$$

and the right hand cohomology group has order $c_{A,\ell}$, which is coprime to p. Thus $\mathrm{res}_\ell(i(x)) = 0$, which completes the sketch of the proof. □

2.5.4 Future Directions

The data in Tables 2.5.1-2.5.4 could be investigated further.

It should be possible to replace the hypothesis that $B[p] \subset A$, with the weaker hypothesis that $B[\mathfrak{m}] \subset A$, where $\mathfrak{m}$ is a maximal ideal of the Hecke algebra $\mathbf{T}$. For example, this improvement would help one to show that 5^2 divides the order of the Shafarevich-Tate group of **1041E**. Note that for this example, we only know that $L(B, 1) = 0$, not that $B(\mathbf{Q})$ has positive rank (as predicted by Conjecture 2.1.5), which is another obstruction.

One can consider visibility at a higher level. For example, there are elements of order 3 in the Shafarevich-Tate group of **551H** that are not visible in $J_0(551)$, but these elements are visible in $J_0(2 \cdot 551)$, according to the computations in [Ste03] (Studying the Birch and Swinnerton-Dyer Conjecture for Modular Abelian Varieties Using MAGMA).

Conjecture 2.5.9 (Stein). *Suppose $c \in Ш(A_f)$, where $A_f \subset J_0(N)$. Then there exists M such that c is visible in $J_0(NM)$. In other words, every element of $Ш(A_f)$ is "modular".*

TABLE 2.5.1. Visibility of Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	odd deg(φ_A)	B	dim	$A \cap B$	Vis
389E$*$	20	5^2	$=$	5	**389A**	1	$[20^2]$	5^2
433D$*$	16	7^2	$=$	$7\cdot{\scriptstyle 111}$	**433A**	1	$[14^2]$	7^2
446F$*$	8	11^2	$=$	$11\cdot{\scriptstyle 359353}$	**446B**	1	$[11^2]$	11^2
551H	18	3^2	$=$	${\scriptstyle 169}$	NONE			
563E$*$	31	13^2	$=$	13	**563A**	1	$[26^2]$	13^2
571D$*$	2	3^2	$=$	$3^2\cdot{\scriptstyle 127}$	**571B**	1	$[3^2]$	3^2
655D$*$	13	3^4	$=$	$3^2\cdot{\scriptstyle 9799079}$	**655A**	1	$[36^2]$	3^4
681B	1	3^2	$=$	$3\cdot{\scriptstyle 125}$	**681C**	1	$[3^2]$	–
707G$*$	15	13^2	$=$	$13\cdot{\scriptstyle 800077}$	**707A**	1	$[13^2]$	13^2
709C$*$	30	11^2	$=$	11	**709A**	1	$[22^2]$	11^2
718F$*$	7	7^2	$=$	$7\cdot{\scriptstyle 5371523}$	**718B**	1	$[7^2]$	7^2
767F	23	3^2	$=$	${\scriptstyle 1}$	NONE			
794G	12	11^2	$=$	$11\cdot{\scriptstyle 34986189}$	**794A**	1	$[11^2]$	–
817E	15	7^2	$=$	$7\cdot{\scriptstyle 79}$	**817A**	1	$[7^2]$	–
959D	24	3^2	$=$	${\scriptstyle 583673}$	NONE			
997H$*$	42	3^4	$=$	3^2	**997B**	1	$[12^2]$	3^2
					997C	1	$[24^2]$	3^2
1001F	3	3^2	$=$	$3^2\cdot{\scriptstyle 1269}$	**1001C**	1	$[3^2]$	–
					91A	1	$[3^2]$	–
1001L	7	7^2	$=$	$7\cdot{\scriptstyle 2029789}$	**1001C**	1	$[7^2]$	–
1041E	4	5^2	$=$	$5^2\cdot{\scriptstyle 13589}$	**1041B**	2	$[5^2]$	–
1041J	13	5^4	$=$	$5^3\cdot{\scriptstyle 21120929983}$	**1041B**	2	$[5^4]$	–
1058D	1	5^2	$=$	$5\cdot{\scriptstyle 483}$	**1058C**	1	$[5^2]$	–
1061D	46	151^2	$=$	$151\cdot{\scriptstyle 10919}$	**1061B**	2	$[2^2 302^2]$	–
1070M	7	$3\cdot 5^2$	$3^2\cdot 5^2$	$3\cdot 5\cdot{\scriptstyle 1720261}$	**1070A**	1	$[15^2]$	–
1077J	15	3^4	$=$	$3^2\cdot{\scriptstyle 1227767047943}$	**1077A**	1	$[9^2]$	–
1091C	62	7^2	$=$	${\scriptstyle 1}$	NONE			
1094F$*$	13	11^2	$=$	$11^2\cdot{\scriptstyle 172446773}$	**1094A**	1	$[11^2]$	11^2
1102K	4	3^2	$=$	$3^2\cdot{\scriptstyle 31009}$	**1102A**	1	$[3^2]$	–
1126F$*$	11	11^2	$=$	$11\cdot{\scriptstyle 13990352759}$	**1126A**	1	$[11^2]$	11^2
1137C	14	3^4	$=$	$3^2\cdot{\scriptstyle 64082807}$	**1137A**	1	$[9^2]$	–
1141I	22	7^2	$=$	$7\cdot{\scriptstyle 528921}$	**1141A**	1	$[14^2]$	–
1147H	23	5^2	$=$	$5\cdot{\scriptstyle 729}$	**1147A**	1	$[10^2]$	–
1171D$*$	53	11^2	$=$	$11\cdot{\scriptstyle 81}$	**1171A**	1	$[44^2]$	11^2
1246B	1	5^2	$=$	$5\cdot{\scriptstyle 81}$	**1246C**	1	$[5^2]$	–
1247D	32	3^2	$=$	$3^2\cdot{\scriptstyle 2399}$	**43A**	1	$[36^2]$	–
1283C	62	5^2	$=$	$5\cdot{\scriptstyle 2419}$	NONE			
1337E	33	3^2	$=$	${\scriptstyle 71}$	NONE			
1339G	30	3^2	$=$	${\scriptstyle 5776049}$	NONE			
1355E	28	3	3^2	$3^2\cdot{\scriptstyle 2224523985405}$	NONE			
1363F	25	31^2	$=$	$31\cdot{\scriptstyle 34889}$	**1363B**	2	$[2^2 62^2]$	–
1429B	64	5^2	$=$	${\scriptstyle 1}$	NONE			
1443G	5	7^2	$=$	$7^2\cdot{\scriptstyle 18525}$	**1443C**	1	$[7^1 14^1]$	–
1446N	7	3^2	$=$	$3\cdot{\scriptstyle 17459029}$	**1446A**	1	$[12^2]$	–

TABLE 2.5.2. Visibility of Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	odd deg(φ_A)	B	dim	$A \cap B$	Vis
1466H∗	23	13^2	=	$13\cdot$25631993723	**1466B**	1	$[26^2]$	13^2
1477C∗	24	13^2	=	$13\cdot$57037637	**1477A**	1	$[13^2]$	13^2
1481C	71	13^2	=	70825	NONE			
1483D∗	67	$3^2\cdot 5^2$	=	$3\cdot 5$	**1483A**	1	$[60^2]$	$3^2\cdot 5^2$
1513F	31	3	3^4	$3\cdot$759709	NONE			
1529D	36	5^2	=	535641763	NONE			
1531D	73	3	3^2	3	**1531A**	1	$[48^2]$	–
1534J	6	3	3^2	$3^2\cdot$635931	**1534B**	1	$[6^2]$	–
1551G	13	3^2	=	$3\cdot$110659885	**141A**	1	$[15^2]$	–
1559B	90	11^2	=	1	NONE			
1567D	69	$7^2\cdot 41^2$	=	$7\cdot 41$	**1567B**	3	$[4^4 1148^2]$	–
1570J∗	6	11^2	=	$11\cdot$228651397	**1570B**	1	$[11^2]$	11^2
1577E	36	3	3^2	$3^2\cdot$15	**83A**	1	$[6^2]$	–
1589D	35	3^2	=	6005292627343	NONE			
1591F∗	35	31^2	=	$31\cdot$2401	**1591A**	1	$[31^2]$	31^2
1594J	17	3^2	=	$3\cdot$259338050025131	**1594A**	1	$[12^2]$	–
1613D∗	75	5^2	=	$5\cdot$19	**1613A**	1	$[20^2]$	5^2
1615J	13	3^4	=	$3^2\cdot$13317421	**1615A**	1	$[9^1 18^1]$	–
1621C∗	70	17^2	=	17	**1621A**	1	$[34^2]$	17^2
1627C∗	73	3^4	=	3^2	**1627A**	1	$[36^2]$	3^4
1631C	37	5^2	=	6354841131	NONE			
1633D	27	$3^6\cdot 7^2$	=	$3^5\cdot 7\cdot$31375	**1633A**	3	$[6^4 42^2]$	–
1634K	12	3^2	=	$3\cdot$3311565989	**817A**	1	$[3^2]$	–
1639G∗	34	17^2	=	$17\cdot$82355	**1639B**	1	$[34^2]$	17^2
1641J∗	24	23^2	=	$23\cdot$1491344147471	**1641B**	1	$[23^2]$	23^2
1642D∗	14	7^2	=	$7\cdot$123398360851	**1642A**	1	$[7^2]$	7^2
1662K	7	11^2	=	$11\cdot$16610917393	**1662A**	1	$[11^2]$	–
1664K	1	5^2	=	$5\cdot$7	**1664N**	1	$[5^2]$	–
1679C	45	11^2	=	6489	NONE			
1689E	28	3^2	=	$3\cdot$172707180029157365	**563A**	1	$[3^2]$	–
1693C	72	1301^2	=	1301	**1693A**	3	$[2^4 2602^2]$	–
1717H∗	34	13^2	=	$13\cdot$345	**1717B**	1	$[26^2]$	13^2
1727E	39	3^2	=	118242943	NONE			
1739F	43	659^2	=	$659\cdot$151291281	**1739C**	2	$[2^2 1318^2]$	–
1745K	33	5^2	=	$5\cdot$1971380677489	**1745D**	1	$[20^2]$	–
1751C	45	5^2	=	$5\cdot$707	**103A**	2	$[505^2]$	–
1781D	44	3^2	=	61541	NONE			
1793G∗	36	23^2	=	$23\cdot$8846589	**1793B**	1	$[23^2]$	23^2
1799D	44	5^2	=	201449	NONE			
1811D	98	31^2	=	1	NONE			
1829E	44	13^2	=	3595	NONE			
1843F	40	3^2	=	8389	NONE			
1847B	98	3^6	=	1	NONE			
1871C	98	19^2	=	14699	NONE			

TABLE 2.5.3. Visibility of Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	odd deg(φ_A)	B	dim	$A \cap B$	Vis
1877B	86	7^2	=	1	NONE			
1887J	12	5^2	=	$5\cdot 10825598693$	**1887A**	1	$[20^2]$	–
1891H	40	7^4	=	$7^2\cdot 44082137$	**1891C**	2	$[4^2 196^2]$	–
1907D∗	90	7^2	=	$7\cdot 165$	**1907A**	1	$[56^2]$	7^2
1909D∗	38	3^4	=	$3^2\cdot 9317$	**1909A**	1	$[18^2]$	3^4
1913B∗	1	3^2	=	$3\cdot 103$	**1913A**	1	$[3^2]$	3^2
1913E	84	$5^4\cdot 61^2$	=	$5^2\cdot 61\cdot 103$	**1913A**	1	$[10^2]$	–
					1913C	2	$[2^2 610^2]$	–
1919D	52	23^2	=	675	NONE			
1927E	45	3^2	3^4	52667	NONE			
1933C	83	$3^2\cdot 7$	$3^2\cdot 7^2$	$3\cdot 7$	**1933A**	1	$[42^2]$	3^2
1943E	46	13^2	=	62931125	NONE			
1945E∗	34	3^2	=	$3\cdot 571255479184807$	**389A**	1	$[3^2]$	3^2
1957E∗	37	$7^2\cdot 11^2$	=	$7\cdot 11\cdot 3481$	**1957A**	1	$[22^2]$	11^2
					1957B	1	$[14^2]$	7^2
1979C	104	19^2	=	55	NONE			
1991C	49	7^2	=	1634403663	NONE			
1994D	26	3	3^2	$3^2\cdot 46197281414642501$	**997B**	1	$[3^2]$	–
1997C	93	17^2	=	1	NONE			
2001L	11	3^2	=	$3^2\cdot 44513447$	NONE			
2006E	1	3^2	=	$3\cdot 805$	**2006D**	1	$[3^2]$	–
2014L	12	3^2	=	$3^2\cdot 126381129003$	**106A**	1	$[9^2]$	–
2021E	50	5^6	=	$5^2\cdot 729$	**2021A**	1	$[100^2]$	5^4
2027C∗	94	29^2	=	29	**2027A**	1	$[58^2]$	29^2
2029C	90	$5^2\cdot 269^2$	=	$5\cdot 269$	**2029A**	2	$[2^2 2690^2]$	–
2031H∗	36	11^2	=	$11\cdot 1014875952355$	**2031C**	1	$[44^2]$	11^2
2035K	16	11^2	=	$11\cdot 218702421$	**2035C**	1	$[11^1 22^1]$	–
2038F	25	5	5^2	$5^2\cdot 92198576587$	**2038A**	1	$[20^2]$	–
					1019B	1	$[5^2]$	–
2039F	99	$3^4\cdot 5^2$	=	13741381043009	NONE			
2041C	43	3^4	=	61889617	NONE			
2045I	39	3^4	=	$3^3\cdot 3123399893$	**2045C**	1	$[18^2]$	–
					409A	13	$[9370199679^2]$	–
2049D	31	3^2	=	29174705448000469937	NONE			
2051D	45	7^2	=	$7\cdot 674652424406369$	**2051A**	1	$[56^2]$	–
2059E	45	$5\cdot 7^2$	$5^2\cdot 7^2$	$5^2\cdot 7\cdot 167359757$	**2059A**	1	$[70^2]$	–
2063C	106	13^2	=	8479	NONE			
2071F	48	13^2	=	36348745	NONE			
2099B	106	3^2	=	1	NONE			
2101F	46	5^2	=	$5\cdot 11521429$	**191A**	2	$[155^2]$	–
2103E	37	$3^2\cdot 11^2$	=	$3^2\cdot 11\cdot 874412923071571792611$	**2103B**	1	$[33^2]$	11^2
2111B	112	211^2	=	1	NONE			
2113B	91	7^2	=	1	NONE			
2117E∗	45	19^2	=	$19\cdot 1078389$	**2117A**	1	$[38^2]$	19^2

TABLE 2.5.4. Visibility of Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	odd deg(φ_A)	B	dim	$A \cap B$	Vis
2119C	48	7^2	=	89746579	NONE			
2127D	34	3^2	=	$3\cdot 18740561792121901$	**709A**	1	$[3^2]$	–
2129B	102	3^2	=	1	NONE			
2130Y	4	7^2	=	$7\cdot 83927$	**2130B**	1	$[14^2]$	–
2131B	101	17^2	=	1	NONE			
2134J	11	3^2	=	1710248025389	NONE			
2146J	10	7^2	=	$7\cdot 1672443$	**2146A**	1	$[7^2]$	–
2159E	57	13^2	=	31154538351	NONE			
2159D	56	3^4	=	233801	NONE			
2161C	98	23^2	=	1	NONE			
2162H	14	3	3^2	$3\cdot 6578391763$	NONE			
2171E	54	13^2	=	271	NONE			
2173H	44	199^2	=	$199\cdot 3581$	**2173D**	2	$[398^2]$	–
2173F	43	19^2	$3^2\cdot 19^2$	$3^2\cdot 19\cdot 229341$	**2173A**	1	$[38^2]$	19^2
2174F	31	5^2	=	$5\cdot 21555702093188316107$	NONE			
2181E	27	7^2	=	$7\cdot 7217996450474835$	**2181A**	1	$[28^2]$	–
2193K	17	3^2	=	$3\cdot 15096035814223$	**129A**	1	$[21^2]$	–
2199C	36	7^2	=	$7^2\cdot 13033437060276603$	NONE			
2213C	101	3^4	=	19	NONE			
2215F	46	13^2	=	$13\cdot 1182141633$	**2215A**	1	$[52^2]$	–
2224R	11	79^2	=	79	**2224G**	2	$[79^2]$	–
2227E	51	11^2	=	259	NONE			
2231D	60	47^2	=	91109	NONE			
2239B	110	11^4	=	1	NONE			
2251E$*$	99	37^2	=	37	**2251A**	1	$[74^2]$	37^2
2253C$*$	27	13^2	=	$13\cdot 14987929400988647$	**2253A**	1	$[26^2]$	13^2
2255J	23	7^2	=	15666366543129	NONE			
2257H	46	$3^6\cdot 29^2$	=	$3^3\cdot 29\cdot 175$	**2257A**	1	$[9^2]$	–
					2257D	2	$[2^2 174^2]$	–
2264J	22	73^2	=	73	**2264B**	2	$[146^2]$	–
2265U	14	7^2	=	$7^2\cdot 73023816368925$	**2265B**	1	$[7^2]$	–
2271I$*$	43	23^2	=	$23\cdot 392918345997771783$	**2271C**	1	$[46^2]$	23^2
2273C	105	7^2	=	7^2	NONE			
2279D	61	13^2	=	96991	NONE			
2279C	58	5^2	=	1777847	NONE			
2285E	45	151^2	=	$151\cdot 138908751161$	**2285A**	2	$[2^2 302^2]$	–
2287B	109	71^2	=	1	NONE			
2291C	52	3^2	=	427943	NONE			
2293C	96	479^2	=	479	**2293A**	2	$[2^2 958^2]$	–
2294F	15	3^2	=	$3\cdot 6289390462793$	**1147A**	1	$[3^2]$	–
2311B	110	5^2	=	1	NONE			
2315I	51	3^2	=	$3\cdot 4475437589723$	**463A**	16	$[13426312769169^2]$	–
2333C	101	83341^2	=	83341	**2333A**	4	$[2^6 166682^2]$	–