Lecture 21: Binary Quadratic Forms I: Sums of Two Squares

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Today we study the question of which integers are the sum of two squares.

1 Sums of Two Squares

During the next four lectures, we will study binary quadratic forms. A simple example of a binary quadratic form that will occupy us today is

$$x^2 + y^2.$$

A typical question that one asks about a quadratic form is which integers does it represent. "Are there integers x and y so that $x^2+y^2 = 389$? So that $x^2+y^2 = 2001$?"

1.1 Which Numbers are the Sum of Two Squares?

The main goal of today's lecture is to prove the following theorem.

Theorem 1.1. *A number n is a sum of two squares if and only if all prime factors of n of the form $4m + 3$ have even exponent in the prime factorization of n.*

Before tackling a proof, we consider a few examples.

Example 1.2.

- $5 = 1^2 + 2^2$.
- 7 is not a sum of two squares.
- 2001 is divisible by 3 because $2 + 1$ is, but not by 9 since $2 + 1$ is not, so 2001 is *not* a sum of two squares.
- $2 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$ is a sum of two squares.
- 389 is a sum of two squares, since $389 \equiv 1 \pmod 4$ and 389 is prime.
- $21 = 3 \cdot 7$ is *not* a sum of two squares even though $21 \equiv 1 \pmod 4$.

In preparation for the proof of Theorem 1.1, we recall a result that emerged when we analyzed how partial convergents of a continued fraction converge.

Lemma 1.3. *If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then there is a fraction $\frac{a}{b}$ in lowest terms such that $0 < b \leq n$ and*

$$\left|x - \frac{a}{b}\right| \leq \frac{1}{b(n+1)}.$$

Proof. Let $[a_0, a_1, \ldots]$ be the continued fraction expansion of x. As we saw in the proof of Theorem 2.3 in Lecture 18, for each m

$$\left|x - \frac{p_m}{q_m}\right| < \frac{1}{q_m \cdot q_{m+1}}.$$

Since q_{m+1} is always at least 1 bigger than q_m and $q_0 = 1$, either there exists an m such that $q_m \leq n < q_{m+1}$, or the continued fraction expansion of x is finite and n is larger than the denominator of the rational number x. In the first case,

$$\left|x - \frac{p_m}{q_m}\right| < \frac{1}{q_m \cdot q_{m+1}} \leq \frac{1}{q_m \cdot (n+1)},$$

so $\frac{a}{b} = \frac{p_m}{q_m}$ satisfies the conclusion of the lemma. In the second case, just let $\frac{a}{b} = x$. $\square$

Definition 1.4. A representation $n = x^2 + y^2$ is *primitive* if $\gcd(x, y) = 1$.

Lemma 1.5. *If n is divisible by a prime p of the form $4m + 3$, then n has no primitive representations.*

Proof. If n has a primitive representation, $n = x^2 + y^2$, then

$$p \mid x^2 + y^2 \quad \text{and} \quad \gcd(x, y) = 1,$$

so $p \nmid x$ and $p \nmid y$. Thus $x^2 + y^2 \equiv 0 \pmod{p}$ so, since $\mathbb{Z}/p\mathbb{Z}$ is a field we can divide by y^2 and see that

$$(x/y)^2 \equiv -1 \pmod{p}.$$

Thus the quadratic residue symbol $\left(\frac{-1}{p}\right)$ equals $+1$. However,

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = (-1)^{\frac{4m+3-1}{2}} = (-1)^{2m+1} = -1.$$

$\square$

Proof of Theorem 1.1. ($\Longrightarrow$) Suppose that p is of the form $4m + 3$, that $p^r \parallel n$ (exactly divides) with r odd, and that $n = x^2 + y^2$. Letting $d = \gcd(x, y)$, we have

$$x = dx', \quad y = dy', \quad n = d^2 n'$$

with $\gcd(x', y') = 1$ and

$$(x')^2 + (y')^2 = n'.$$

Because r is odd, $p \mid n'$, so Lemma 1.5 implies that $\gcd(x', y') > 1$, a contradiction.

($\Longleftarrow$) Write $n = n_1^2 n_2$ where n_2 has no prime factors of the form $4m + 3$. It suffices to show that n_2 is a sum of two squares. Also note that

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1x_2 + y_1y_2)^2 + (x_1y_2 - x_2y_1)^2,$$

so a product of two numbers that are sums of two squares is also a sum of two squares.[1] Also, the prime 2 is a sum of two squares. It thus suffices to show that if p is a prime of the form $4m + 1$, then p is a sum of two squares.

Since

$$(-1)^{\frac{p-1}{2}} = (-1)^{\frac{4m+1-1}{2}} = +1,$$

-1 is a square modulo p; i.e., there exists r such that $r^2 \equiv -1 \pmod{p}$. Taking $n = \lfloor\sqrt{p}\rfloor$ in Lemma 1.3 we see that there are integers a, b such that $0 < b < \sqrt{p}$ and

$$\left|-\frac{r}{p} - \frac{a}{b}\right| \le \frac{1}{b(n+1)} < \frac{1}{b\sqrt{p}}.$$

If we write

$$c = rb + pa$$

then

$$|c| < \frac{pb}{b\sqrt{p}} = \frac{p}{\sqrt{p}} = \sqrt{p}$$

and

$$0 < b^2 + c^2 < 2p.$$

But $c \equiv rb \pmod{p}$, so

$$b^2 + c^2 \equiv b^2 + r^2b^2 \equiv b^2(1 + r^2) \equiv 0 \pmod{p}.$$

Thus $b^2 + c^2 = p$. $\square$

1.2 Computing x and y

Suppose p is a prime of the form $4m + 1$. There is a construction of Legendre of x and y that is explained on pages 120–121 of Davenport. I'm unconvinced that it is any more efficient than the following naive algorithm: compute $\sqrt{p - x^2}$ for $x = 1, 2, \ldots$ until it's an integer. This takes at most $\sqrt{p}$ steps. Here's a simple PARI program which implements this algorithm.

[1] This algebraic identity is secretely the assertion that the norm map $N : \mathbb{Q}(i)^* \to \mathbb{Q}^*$ sending $x + iy$ to $(x + iy)(x - iy) = x^2 + y^2$ is a homomorphism.

```
{sumoftwosquares(n) =
   local(y);
   for(x=1,floor(sqrt(n)),
      y=sqrt(n-x^2);
      if(y-floor(y)==0, return([x,floor(y)]))
   );
   error(n," is not a sum of two squares.")
}
```

2 Sums of More Squares

Every natural number is a sum of **four** squares. See pages 124–126 of Davenport for a proof.

A natural number is a sum of **three** squares if and only if it is not a power of 4 times a number that is congruent to 7 modulo 8. For example, 7 is not a sum of three squares. This is more difficult to prove.