Lecture 19: Continued Fractions III
Quadratic Irrationals

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In this lecture we prove that the continued fraction expansion of a number is periodic if and only if the number is a quadratic irrational.

1 Quadratic Irrationals

Definition 1.1. An element $\alpha \in \mathbb{R}$ is a *quadratic irrational* if it is irrational and satisfies a quadratic polynomial.

Thus, e.g., $(1+\sqrt{5})/2$ is a quadratic irrational. Recall that

$$\frac{1+\sqrt{5}}{2} = [1, 1, 1, \ldots].$$

The continued fraction of $\sqrt{2}$ is $[1, 2, 2, 2, 2, 2, \ldots]$, and the continued fraction of $\sqrt{389}$ is

$$[19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \ldots].$$

Does the $[1, 2, 1, 1, 1, 1, 2, 1, 38]$ pattern repeat over and over again??

2 Periodic Continued Fractions

Definition 2.1. A *periodic continued fraction* is a continued fraction $[a_0, a_1, \ldots, a_n, \ldots]$ such that

$$a_n = a_{n+h}$$

for a fixed positive integer h and all sufficiently large n. We call h the *period* of the continued fraction.

Example 2.2. Consider the periodic continued fraction $[1, 2, 1, 2, \ldots] = [\overline{1, 2}]$. What does it converge to?

$$[\overline{1,2}] = 1 + \cfrac{1}{2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cdots}}}},$$

so if $\alpha = [\overline{1, 2}]$ then

$$\alpha = 1 + \frac{1}{2 + \alpha}.$$

Thus $2\alpha + \alpha^2 = 2 + \alpha + 1$, so

$$\alpha^2 + \alpha - 3 = 0 \quad \text{and} \quad \alpha = \frac{-1 + \sqrt{7}}{2}.$$

Theorem 2.3. *An infinite integral continued fraction is periodic if and only if it represents a quadratic irrational.*

Proof. $(\Longrightarrow)$ First suppose that

$$[a_0, a_1, \ldots, a_n, \overline{a_{n+1}, \ldots, a_{n+h}}]$$

is a periodic continued fraction. Set $\alpha = [a_{n+1}, a_{n+2}, \ldots]$. Then

$$\alpha = [a_{n+1}, \ldots, a_{n+h}, \alpha],$$

so

$$\alpha = \frac{\alpha p_{n+h} + p_{n+h-1}}{\alpha q_{n+h} + q_{n+h-1}}.$$

(We use that α is the last partial convergent.) Thus α satisfies a quadratic equation. Since the a_i are all integers, the number

$$\begin{aligned} [a_0, a_1, \ldots] &= [a_0, a_1, \ldots, a_n, \alpha] \\ &= a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \alpha}} \end{aligned}$$

can be expressed as a polynomial in α with rational coefficients, so $[a_0, a_1, \ldots]$ also satisfies a quadratic polynomial. Finally, $\alpha \notin \mathbb{Q}$ because periodic continued fractions have infinitely many terms.

$(\Longleftarrow)$ This direction was first proved by Lagrange. The proof is much more exciting! Suppose $\alpha \in \mathbb{R}$ satisfies a quadratic equation

$$a\alpha^2 + b\alpha + c = 0$$

with $a, b, c \in \mathbb{Z}$. Let $[a_0, a_1, \ldots]$ be the expansion of α. For each n, let

$$r_n = [a_n, a_{n+1}, \ldots],$$

so that

$$\alpha = [a_0, a_1, \ldots, a_{n-1}, r_n].$$

We have

$$\alpha = \frac{r_n p_n + p_{n-1}}{r_n q_n + q_{n-1}}.$$

Substituting this expression for α into the quadratic equation for α, we see that

$$A_n r_n^2 + B_n r_n + C_n = 0,$$

where

$$\begin{aligned}
A_n &= a p_{n-1}^2 + b p_{n-1} q_{n-1} + c q_{n-1}^2, \\
B_n &= 2a p_{n-1} p_{n-2} + b(p_{n-1} q_{n-2} + p_{n-2} q_{n-1}) + 2c q_{n-1} q_{n-2}, \\
C_n &= a p_{n-2}^2 + b p_{n-2} q_{n-2} + c p_{n-2}^2.
\end{aligned}$$

Note that $A_n, B_n, C_n \in \mathbb{Z}$, that $C_n = A_{n-1}$, and that

$$B^2 - 4A_n C_n = (b^2 - 4ac)(p_{n-1} q_{n-2} - q_{n-1} p_{n-2})^2 = b^2 - 4ac.$$

Recall from the proof of Theorem 2.3 of the previous lecture that

$$\left| \alpha - \frac{p_{n-1}}{q_{n-1}} \right| < \frac{1}{q_n q_{n-1}}.$$

Thus

$$|\alpha q_{n-1} - p_{n-1}| < \frac{1}{q_n} < \frac{1}{q_{n+1}},$$

so

$$p_{n-1} = \alpha q_{n-1} + \frac{\delta}{q_{n-1}} \qquad \text{with } |\delta| < 1.$$

Hence

$$\begin{aligned}
A_n &= a \left(\alpha q_{n-1} + \frac{\delta}{q_{n-1}} \right)^2 + b \left(\alpha q_{n-1} + \frac{\delta}{q_{n-1}} \right) q_{n-1} + c q_{n-1}^2 \\
&= (a\alpha^2 + b\alpha + c) q_{n-1}^2 + 2a\alpha\delta + a \frac{\delta^2}{q_{n-1}^2} + b\delta \\
&= 2a\alpha\delta + a \frac{\delta^2}{q_{n-1}^2} + b\delta.
\end{aligned}$$

Thus

$$|A_n| = \left| 2a\alpha\delta + a \frac{\delta^2}{q_{n-1}^2} + b\delta \right| < 2|a\alpha| + |a| + |b|.$$

Thus there are only finitely many possibilities for the integer A_n. Also,

$$|C_n| = |A_{n-1}| \quad \text{and} \quad |B_n| = \sqrt{b^2 - 4(ac - A_n C_n)},$$

so there are only finitely many triples (A_n, B_n, C_n), and hence only finitely many possibilities for r_n as n varies. Thus for some $h > 0$,

$$r_n = r_{n+h}.$$

This shows that the continued fraction for α is periodic. □

3 What About Higher Degree?

Definition 3.1. An *algebraic number* is a root of a polynomial $f \in \mathbb{Q}[x]$.

Open Problem: [1] What is the continued fraction expansion of the algebraic num-

[1] As far as I know this is still an open problem.

ber $\sqrt[3]{2}$?

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? contfrac(2^(1/3))
%5 = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3,
 2, 1, 3, 4, 1, 1, 2, 14, 3, 12, 1, 15, 3, 1, 4, 534, 1, 1, 5, 1, 1,
 121, 1, 2, 2, 4, 10, 3, 2, 2, 41, 1, 1, 1, 3, 7, 2, 2, 9, 4, 1, 3, 7,
 6, 1, 1, 2, 2, 9, 3, 1, 1, 69, 4, 4, 5, 12, 1, 1, 5, 15, 1, 4, 1, 1,
 1, 1, 1, 89, 1, 22, 186, 5, 2, 4, 3, 3, 1, \ldots]
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I sure don't see a pattern, and that 534 strips me of any confidence that I ever will. One could at least try to analyze the first few terms of the continued fraction statistically (see Lang and Trotter, 1972).

Khintchine (1963), page 59:

> No properties of the representing continued fractions, analogous to those which have just been proved, are known for algebraic numbers of higher degree. [...] It is of interest to point out that up till the present time no continued fraction development of an algebraic number of higher degree than the second is known. It is not even known if such a development has bounded elements. Generally speaking the problems associated with the continued fraction expansion of algebraic numbers of degree higher than the second are extremely difficult and virtually unstudied.

Richard Guy *Unsolved Problems in Number Theory* (1994), page 260:

> Is there an algebraic number of degree greater than two whose simple continued fraction has unbounded partial quotients? Does *every* such number have unbounded partial quotients?