A. Student

Math 124 Problem Set 7

1. D=-155 There are four elements: $[[1,1,39],[3,-1,13],[3,1,13],[5,5,9]]$.

By the structure theorem, $\mathcal{C}_{-155}$ is isomorphic to either $C_2\text{x}C_2$ or C_4. It is easy to verify that $[1,1,39]$ is the identity. From this we find that $[3,-1,13]$ has order 4, so it must be that $\mathcal{C}_{-155} \backsimeq C_4$.

D=-231 There are twelve elements: $[1,1,58],[2,-1,29],[2,1,29],[3,3,20],[4,-3,15],[4,3,15]$, $[5,-3,12],[5,3,12],[6,-3,10],[6,3,10],[7,7,10],[8,5,8]$. Therefore $\mathcal{C}_{-231} \simeq C_{12}$ or $C_2\text{x}C_6$. The identity is $[1,1,58]$. Both $[2,-1,29]$ and $[2,1,29]$ have order 6, which is impossible in C_{12}, so $\mathcal{C}_{-231} \simeq C_2\text{x}C_6$.

D=-660 There are eight elements: $[1,0,165],[10,10,19],[11,0,15],[13,4,13],[2,2,83],[3,0,55],[5,0,33]$, $[6,6,29]$. The first element is the identity, and all others have order 2. Therefore $\mathcal{C}_{-660} \simeq C_2\text{x}C_2\text{x}C_2$.

D=-12104 There are forty-eight elements: (listed in an email from Professor Stein). By the structure theorem, $\mathcal{C}_D \simeq C_{48}$, $C_4\text{x}C_{12}$, or $C_2\text{x}C_{24}$. The identity element is $[1,0,3026]$, and using it we find two elements of order four: $[45,-26,71]$ and $[50,-36,67]$, eliminating everything but $C_4\text{x}C_{12}$.

D=-10015 There are fifty-four elements (listed in an email from Professor Stein). Therefore $\mathcal{C}_D \simeq C_3\text{x}C_{18}$ or C_{54}. The identity is $[1,1,2504]$; from this we find two elements with order 9: $[10,-5,251]$ and $[10,5,251]$. Therefore the group cannot be C_{54}, so $\mathcal{C}_D \simeq C_3\text{x}C_{18}$.

2. The three graphs are on the next page, plotted in MAPLE.

3. Differentiating implicitly, the slope of the tangent at (x,y) is $\frac{3x^2}{2y}$. At $(3,5)$, the slope is $\frac{27}{10}$, and the tangent line has equation $y = \frac{27x-31}{10}$. Substituting into the relation $y^2 - x^3 = -2$, we have $(\frac{27x-31}{10})^2 - x^3 = -2$, which simplifies to the polynomial

$$100x^3 - 729x^2 + 1674x - 1161 = 0.$$

This polynomial has a double root at $x = 3$, so it factors into $(x-3)^2(100x-129)$, giving a rational root with $x = 1.29$. Therefore $(1.29, .383)$ is a rational solution to the original equation.