Homework 6: Binary Quadratic Forms

Due Wednesday, November 7

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Math 124 Harvard University **Fall 2001**

There are 9 problems. Work in groups and use PARI as much as you like.

1. (3 points) Which of the following numbers is a sum of two squares? Express those that are as a sum of two squares.

$$-389,\ 12345,\ 91210,\ 729,\ 1729,\ 68252$$

2. (i) (4 points) Write a PARI program that takes a positive integer n as input and outputs a sequence `[x,y,z,w]` of integers such that $x^2 + y^2 + z^2 + w^2 = n$. (Hint: Your program does not have to be efficient.)

   (ii) (2 point) Write 2001 as a sum of three squares.

3. (3 points) Find a positive integer that has a least three different representations as the sum of two squares, disregarding signs and the order of the summands.

4. (5 points) Show that a natural number n is the sum of two integer squares if and only if it is the sum of two rational squares.

5. (6 points) Mimic the proof of the main theorem of Lecture 21 to show that an odd prime p is of the form $8m + 1$ or $8m + 3$ if and only if it can be written as $p = x^2 + 2y^2$ for some choice of integers x and y. (Hint: Use the formula for the quadratic residue symbol $\left(\frac{-2}{p}\right)$ from Lecture 13.)

6. (4 points) A *triangular number* is a number that is the sum of the first m integers for some positive integer m. If n is a triangular number, show that all three of the integers $8n^2$, $8n^2 + 1$, and $8n^2 + 2$ can be written as a sum of two squares.

7. (3 points) Prove that of any four consecutive integers, at least one is not representable as a sum of two squares.

8. (4 points) Show that $13x^2+36xy+25y^2$ and $58x^2+82xy+29y^2$ are each equivalent to the form x^2+y^2, then find integers x and y such that $13x^2+36xy+25y^2 = 389$.

9. (4 points) What are the discriminants of the forms $199x^2 - 162xy + 33y^2$ and $35x^2 - 96xy + 66y^2$? Are these forms equivalent?