Homework 5: Continued Fractions

Due Wednesday, October 31 (Halloween)

William Stein

Math 124 Harvard University **Fall 2001**

There are 10 problems. Feel free to use a computer on any of them.

1. (3 points) Draw some sort of diagram that illustrates the partial convergents of the following continued fractions:

   (i) $[13, 1, 8, 3]$

   (ii) $[1, 1, 1, 1, 1, 1, 1, 1]$

   (iii) $[1, 2, 3, 4, 5, 6, 7, 8]$

2. (5 points) If $c_n = p_n/q_n$ is the nth convergent of the continued fraction $[a_0, a_1, \ldots, a_n]$ and $a_0 > 0$, show that

$$[a_n, a_{n-1}, \ldots, a_1, a_0] = \frac{p_n}{p_{n-1}}$$

   and

$$[a_n, a_{n-1}, \ldots, a_2, a_1] = \frac{q_n}{q_{n-1}}.$$

   (Hint: In the first case, notice that $\frac{p_n}{p_{n-1}} = a_n + \frac{p_{n-2}}{p_{n-1}} = a_n + \frac{1}{\frac{p_{n-1}}{p_{n-2}}}$.)

3. (4 points) There is a function $j(\tau)$, denoted by `ellj` in PARI, which takes as input a complex number τ with positive imaginary part, and returns a complex number called the "j-invariant of the associated elliptic curve". Suppose that τ is *approximately* $-0.5 + 0.3281996289i$ and that you know that $j = j(\tau)$ is a rational number. Use continued fractions and PARI to compute a reasonable guess for the rational number $j = \texttt{ellj}(\tau)$. (Hint: In PARI $\sqrt{-1}$ is represented by `I`.)

4. (3 points) Evaluate each of the following infinite continued fractions:

   (i) $[\overline{2, 3}]$

   (ii) $[2, \overline{1, 2, 1}]$

   (iii) $[0, \overline{1, 2, 3}]$

5. (3 points) Determine the infinite continued fraction of each of the following numbers:

   (i) $\sqrt{5}$

   (ii) $\dfrac{1 + \sqrt{13}}{2}$

   (iii) $\dfrac{5 + \sqrt{37}}{4}$

6. (i) (4 points) For any positive integer n, prove that $\sqrt{n^2+1} = [n, \overline{2n}]$.

   (ii) (2 points) Find a convergent to $\sqrt{5}$ that approximates $\sqrt{5}$ to within four decimal places.

7. (4 points) A famous theorem of Hurwitz (1891) says that for any irrational number x, there exists infinitely many rational numbers a/b such that

$$\left| x - \frac{a}{b} \right| < \frac{1}{\sqrt{5} b^2}.$$

   Taking $x = \pi$, obtain three rational numbers that satisfy this inequality.

8. (3 points) The continued fraction expansion of e is

$$[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots].$$

   It is a theorem that the obvious pattern continues indefinitely. Do you think that the continued fraction expansion of e^2 also exhibits a nice pattern? If so, what do you think it is?

9. (i) (4 points) Show that there are infinitely many even integers n with the property that both $n+1$ and $\frac{n}{2}+1$ are perfect squares.

   (ii) (3 points) Exhibit two such integers that are greater than 389.

10. (7 points) A primitive Pythagorean triple is a triple x, y, z of integers such that $x^2 + y^2 = z^2$. Prove that there exists infinitely many primitive Pythagorean triples x, y, z in which x and y are consecutive integers.