Final Project:

Mathematics behind Card Tricks

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Introduction: The nature of our project was to investigate the underlying mathematics of certain card tricks. Many card tricks employ a slight of hand technique or some other method to deceive a participant by sneaking certain cards into certain parts of a deck, hiding certain cards, and so on. These types of tricks are exciting to observe, but tend to lack mathematical substance. On the other hand, there are many tricks which are built strictly upon mathematics to ensure their practical success. We focused on this type. For the most part, the math involved in the following tricks is elementary. This makes sense, as when tricks become too complicated or long-winded, they quickly become boring.

Note: we would like to apologize for the unprofessional nature of our video demonstrations. Many factors contributed to the subpar theatrics of the actors.

1 Last Two Cards Match

Goal: Start with 2 rows each of 5 cards with the property that the cards in each column will have the same face value. Perform some "random" rotations on the cards of each row, but the column property will remain fixed.

Video: `http://becublog.com/math414/#card1` (watch first)

How to perform: This is a self-working trick. You don't have to know anything other than what was done in the video. But if you're interested in the Math behind it, go on.

Math: In each row we can use any set of 5 cards of any suit (but the column property must hold), but for simplicity we use cards from 1 to 5 as in the video. We can consider each row as a cyclic group of order 5. Notice how both the performer and the spectator collected cards from their left to right, which if combined together creates a line of numbers: 5 4 3 2 1 and 1 2 3 4 5. Before any spelling occurs, the gap between any 2 cards of the same face value is 4
i.e. $x \cong y \pmod 4$.

Claim: After each spelled word, the top cards of the 2 rows have the same face value.

Proof: Consider the ordering: n, n-1, ..., 1 and 1, ..., n-1, n. WLOG, we can assume each spelling round only goes back and forth once, and the spectator always goes first. Suppose the spectator spells k letters, so we have the new order for the first half of the ordering above: n-k, ..., 1, n, ..., k. The performer is left with n-1-k letters, so the other half is 1+(n-1-k), ..., 1, ..., n-1-k = n-k, ..., 1, ..., n-1-k. So after the first spelling round, both rows have n-1 as the face value of the top card. Remove this top card from each row, and we are left with a similar problem (with 1 less card on each row). QED.

Above proof explains why in the next spelling round we need to have a word of 3 letters (in this case "Two") to fill the gap between 4 cards. Next round we are left with 3 cards on each row and the gap is 2; any word of 2 letters would work just fine, but also anything $\cong 2 \pmod 3$ (which explains "Cards").

Discussion: There is nothing special about the number 5, as the above problem works for any positive integer n. Also our demonstration is the simplest case because it turns out that if the performer first collects all 10 cards, does some random cuts, and then creates 2 piles of 5 cards, it will still work (but one has to pay attention when creating 2 piles of cards: i.e. one might get 1 2 3 4 5 and 1 2 3 4 5 instead of 5 4 3 2 1 and 1 2 3 4 5). The result is of course more convincing because now each row might contain more than one suit.

Claim: In the pile 5 4 3 2 1 and 1 2 3 4 5, random cuts don't matter.

Proof: This is obvious because cuts won't affect the relative position (i.e. gap) of a card compared to all other cards in the group. It simply is a rotation of the ordering of all elements in the group. Same for general n. QED

2 Numbers Trick

Goal: Spectator chooses two cards and the magician asks him/her to do some simple arithmetic, after which the magician reveals the participant's chosen cards.

Video: `http://becublog.com/math414/#card2` (watch first)

How to perform: The "Number's Trick" is as simple as they come. But, nevertheless, it is interesting how it employs number theory, albeit at its most rudimentary. The trick is simple to perform. Take a deck of 52 cards and remove all the 10's through Aces. Shuffle the deck and have a victim choose 2 cards at random. Have the person take the value of one of their chosen cards, double this value, add 5 to this value, multiply this value by 5, and then add the value of their second card. Have them tell you the resultant number. Subtract 25 from this number and you have a two digit number, each digit being the value of your participant's cards.

Math: The magician makes sure that the deck has only 1's (Aces) through 9's. This ensures that a number can be easily constructed in base-10 from the participant's chosen cards. The victim picks his/her two cards and simple arithmetic is performed: Let v_1 and v_2 denote the values of the two cards the victim chooses. Then the victim performs $(2v_1+5)*5+v_2 = 10v_1+v_2+25$. The victim then tells the magician the resulting number. The magician then subtracts 25 from this number, which yields $10v_1 + v_2$ which is the number $(v_1v_2)_{10}$. In other words, the victim's cards are the digits of the number the magician computes.

Discussion: This trick is quite trivial. The arithmetic manipulations involved can be simplified or made more tedious at the magician's discretion. We of course want the coefficient of the first number 10 and the second number untouched. Any other addition and multiplication is fine, as long as you don't get lost. For example, we could've said, "multiply your number by 10, add 38,769, add your second number." You would then subtract 38,769. Or, "multiply your number by 5, add a billion, multiply this by 2 and add your second number." You would then subtract 2 billion.

3 Symmetry, take it easy

Goal: Start out with 3 cards (in the video we used 1 (Ace), 2, and 3 of hearts) face up. Have a spectator choose a card behind your back. Switch the positions of the other two cards, then face them all down. The magician then turns around and asks the spectator to turn each card over and reveal his/her choice.

Video: `http://becublog.com/math414/#card3` (watch first)

How to perform: Label 3 cards from left to right (of magician) as number 1, 2, and 3 (in the video it happens to be the same as the 3 cards being used for the trick). After the spectator has done his/her part, pick up the cards from left to right. So the leftmost card will be on top, the middle card is in the middle, and the rightmost card will be on the bottom. Do some random cuts (note that performer in the video mistakenly says "shuffle", but actually he's doing cuts; shuffling will mess up the trick, and we will see why later), but the total number of cards being cut must be $\cong 1 \pmod 3$ (so 4, 7, 10,...cuts only). Deal the cards onto the table as follows: place the first card on the table, second card to the right of first card, and third card to the left of first card. As the spectator turns each card over, if the face value of a card matches the label we gave it earlier, then thats his/her choice.

Math: This trick takes advantage of properties of symmetric group S_3.

Definition: A transposition is a map that swaps two elements of a set.

Originally, we have a permutation which sends $1 \to 2$, $2 \to 3$, and $3 \to 1$. When the spectator switches positions of 2 cards he/she actually creates some transposition, say τ. As you've seen in the video, the key to knowing the spectator's chosen card is to fix it and switch the positions of the other two cards. So no matter what transposition τ is made by spectator, we need to return exactly that transposition at the end. In other words, (permutations made after τ)(τ) = (identity)(τ). Hence other permutations = identity map. Since you pick up the cards from left to right, you create a 3-cycle, and WLOG, we can assume it is (1 3 2). Next, since the magician is required to do some cuts $\cong 1 \pmod 3$, this simply creates a 3-cycle (3 2 1) = (1 3 2). In the end, the way the performer deals the cards onto table creates another 3-cycle (1 3 2).

Lemma: If τ in S_n is a k-cycle, then the order of τ is k, that is $\tau^k = e$.

<u>Proof</u>: The proof can be found in Abstract Algebra by I.N. Herstein (or any algebra book). But it is intuitively clear that if we compose, for example, (1 3 2) with itself 3 times, we get $(132)^3 = (132)(132)(132) = e$. QED.

So we have $(132)(132)(132)(\tau) = (e)(\tau) = (\tau)$. And to this point we should know why the magician must do cuts instead of shuffles: shuffles change the relative positions of cards while cuts don't. So after all the work, we end up with the original card positions. Hence the only fixed card is that which the spectator chose.

Discussion: There is nothing special about 1 (again, Ace), 2, or 3 of hearts. We can choose any 3 cards of any suit, all we need to remember is its position on the table. This trick can also be extended to more than 3 cards, but it becomes a bit trickier because we want the spectator's chosen card to be the only fixed after all permutations. Also it doesn't have to be the same permutation (in above example we had (1 3 2)), as there are many other ways for us to reach the identity map, say $(12)(132)(13) = e$. This can be done by collecting the cards from right to left, performing cuts $\cong 1$ (3), and then dealing in the order middle, right, left. Also, one must note that the original numbering of the cards must now be reversed. Verification is left for the reader.

4 Amazing/Semi-Amazing Math Card Trick

Goal: Spectator chooses a card, puts it back into the deck, and the magician then successfully reveals his/her chosen card.

Video: `http://becublog.com/math414/#card4` (watch first)

How to perform: Have someone chose a random card from a deck of 52. Have the person remember the card and then have him/her place it on top of the bottom 8 cards from the deck. Place the other cards on top and then begin flipping over cards from the top of the deck into a pile. As you flip over the first card begin to count down from 10. So the first card flipped over is "10", the second is "9", and so on, counting all the way down to "1". Now if at any time the card that you flip over matches your count (i.e. you flip over an 8 as you count "8"), then you stop flipping over cards and move the pile to the side. Note that Jacks, Queens, and Kings have a value of 10, and Aces 1. If you count all the way down from 10 to 1 (i.e. no cards match their count) then place a card face down on top of the pile and place it to the side. Start a new pile by the exact same procedure. Make 4 piles in this manner. After 4 piles have been made, observe the top card in each pile, which is either face up or face down. Count all the face up cards. This total will be the position (from the top of the remaining deck) of the card chosen by your participant.

Math: For the "Amazing Math Card Trick" the magician has his/her victim place their chosen card on top of the bottom 8 cards in the deck. This means that the victim's card is the 9th card from the bottom, or, more importantly, the 44th card from the top. The procedure of the trick creates 4 piles of cards, each pile consisting at most of 11 cards (in the case that the face value of a card never matches its position in the count). Hence, there will be at most 44 cards removed from the top of deck, preventing the magician from removing the victim's card.

Now the mathematics of the "stop when a card matches its count position" is as such: Let m_i denote the "match" value in the i-th pile, where $m_i = 0$ if there is no match in a pile. For example, say the first pile has 8 as a match, the second no match, the third 1 as a match, and the fourth pile 3 as a match. Then $m_1 = 8, m_2 = 0, m_3 = 1$, and $m_4 = 3$. Whatever the match may be, each pile then consists of $11 - m_i$ cards. Summing up the four piles yields

$$\sum_{i=1}^{4} 11 - m_i$$

Now since the victim's card is the 44th from the top of the original pile, after the creation of each new pile it is in position $44 - (11 - m_i)$ from the top. Hence, after the creation of 4 piles, the victim's card is in position

$$44 - \sum_{i=1}^{4} 11 - m_i$$

from the top. But,

$$44 - \sum_{i=1}^{4} 11 - m_i = \sum_{i=1}^{4} m_i$$

5 Mother of All Card Tricks

Goal: Start with a "random" shuffled deck. Make some random cuts, then tell the face and suit of the card right after the cut. Moreover, you can tell exactly where in the deck a card is located.

Video: `http://becublog.com/math414/#card5` (watch first)

How to perform: The deck is not really random but looks like a random shuffled deck of cards. Arrange cards in the deck as follows:

A	K	Q	J	10	9	8	7	6	5	4	3	2	(all clubs)
4	3	2	A	K	Q	J	10	9	8	7	6	5	(all hearts)
7	6	5	4	3	2	A	K	Q	J	10	9	8	(all spades)
10	9	8	7	6	5	4	3	2	A	K	Q	J	(all diamonds)

There are 13 columns, each column consisting of 4 cards. Collect cards in each column (top row $\rightarrow$ bottom row) then go to next column. Consider J as 11, Q as 12, and K as 13 in face value. It is clear that the face value of any card is equal to the face value of its preceding card + 3 (mod13). Appropriate suit values: clubs as 1, hearts as 2, spades as 3, and diamonds as 0. Having all these conditions, you are now able to tell the top card (face and suit) after each cut. You can also tell the position of any card in the deck. First you must know the bottom card of the deck–say it has face value b_F and suit value b_S. Suppose the card you want to find has face value n_F and suit value n_S. Its position can then be found using the formula $13(n_S - b_S) - 4(n_F - b_F)$ (mod 52). This also works if you know the top card, an equivalent formula for which being left for the reader to derive.

Math: We're working in modulo 13 for face values (so K $\cong$ 0 mod 13) and modulo 4 for suit values (so diamonds $\cong$ 0 mod 4). So there are 2 different ways to find the face value (n_F) of the n^{th} card from the top:

1. $n_F = t_F + 3$(n-1) (mod 13) where t_F = face value of top card

2. $n_F = b_F + 3$n (mod 13) where b_F = face value of bottom card

Similarly there are 2 different ways to find a suit value (n_S) (hence actual suit) of the n^{th} card from the top:

1. $n_S = t_S +$ (n-1) (mod 4) where t_S = suit value of top card

2. $n_S = b_F +$ n (mod 4) where b_S = suit value of bottom card

WLOG, we can choose the second formula for both face and suit value of the n^{th} card to show the Math for the last trick where the magician uses his ipod to generate a random face value and suit value. He is then able to tell where it is in the stack. We have $n_F = b_F + 3$n (mod 13) $\Rightarrow$ n = -4(n_F - b_F) (mod 13). On the other hand, we also have n = n_S - b_S (mod 4). Since gcd(13, 4) = 1, we can use Chinese Remainder Theorem to find an n that satisfies both equations. This can be done by solving for t in the equation -4(n_F - b_F) + $t * 13$ = (n_S - b_S) (mod 4). We can easily see that t = (n_S - b_S) works. So n = -4(n_F - b_F) + 13(n_S - b_S) (mod 52) which is the formula we gave earlier in the How to Perform section. The fact that n is unique modulo 52 is given by Chinese Remainder Theorem.

6 Final Three Card Trick

Goal: Spectator chooses three cards, places the cards into three piles, and the magician then collects the piles and repeatedly deals out all cards in the deck until there are only 3 cards left, which happen to be participant's cards.

Video: `http://becublog.com/math414/#card6` (watch first)

How to perform: For this trick you have someone choose 3 random cards from a deck of 52. Once he/she has chosen 3 cards, you make three piles from your right to left. The first pile consists of 10 cards, the second, pile 15, and the third pile 15. There will be 9 cards remaining which you place off to the side. Have your contestant place any one of their 3 cards on top of the first pile and then cut the second pile in any way and place the cut on top of the first pile. Then have them place one of their two remaining cards on top of the second pile (which has been cut) and place this pile on top of the first. Finally, have them place their last card on top of the third pile, and then place the 9 cards off to the side on top of this pile. Place this pile on top of the other pile. With the deck all in place, take four cards from the top and move them to the bottom of the deck. Then flip over the top card face up. Then flip over the next card face down next to the face up card. Alternate in this manner until the deck is exhausted. Discard the face up cards and begin this procedure again with the pile of face down cards. Continue until there are only 3 face down cards left, which will be you participant's cards.

Math: For the "Final Three Card Trick" the positions of the 3 cards are critical. The procedure of this trick involves creating two piles by alternately placing one card face up, the next card face down, continuing until the deck is exhausted, and then discarding the face-up pile. If one assigns positions to the cards in a deck of 52, starting from bottom to top, then we have: $(1, 2, 3, ..., 52)$. And so after performing the first "up-down" procedure we have 26 cards, and, relative to their initial positions in the original deck, are arranged as so: $(51, 49, 47, ..., 1)$. In other words, the card that was the 51st card from the bottom in the original deck is now the card at the very bottom, and so forth. Now performing the procedure again, we have 13 cards with the ordering $(3, 7, 11, ..., 51)$. Performing again, we have $(47, 39, 31, 23, 15, 7)$. Performing the last "up-down" yields $(15, 31, 47)$. Hence, if one is to start with a deck of 52 and continually remove half the cards in this manner and stop at 3 cards, the only three cards that will remain are the cards 15th, 31st, and 47th from the bottom of the original deck. It should also be noted that when the deck reduces to 13 cards, the pile of the first card to be placed down (the face up pile) will contain the odd amount of cards, leaving the nice even number 6 to divide in half and provide us with our final three cards. So discarding the face up pile is not trivial.

Now the size of the first pile in the beginning setup can be modified, but only slightly. 10 is arguably the most aesthetically pleasing number, and this is why it is used. With the original set up of the trick (piles of sizes 10, 15, and 15, respectively), after the victim's cards are back in the deck, his/her cards will be in positions $(11, 27, 43)$. This is why 4 cards are removed from the top and placed on the bottom, which is in essence the permutation σ^4, where $\sigma = (1, 2, 3, ..., 52)$. Now since we must have our victim's three cards in positions (15,31,47) the first pile we create in the beginning can be no larger than 14 (if we are to move cards from the top to the bottom to cycle the cards into positions (15,31,47); else we must move cards from the top of the deck to the bottom to cycle the deck which is redundant). Now whatever the size of the first pile may be, we must have that the number of positions between the first and second card of our victim, as well as the second and third card, within the deck is exactly 16 positions. Hence, we must have that the second and third pile consist of exactly 15 cards each (since we must account for each card the victim places on each pile). For example, after the victim chooses their cards, if we set up the first pile with 6 cards, we must set up the other two with 15 cards, and then after the victim's cards are within the deck, we move 8 cards from the top of the deck to the bottom so that their three cards go from positions (7,23,39) to (15,31,47). It's nothing fancy. (15, 31,47) are the magic numbers.

7 The Australian Shuffle

Goal: Spectator chooses a random number between 10 and 30. Performer then deals out that many cards. Performer then does the Austrialian shuffle until there is exactly one card left. Performer then correctly names that card (face and suit).

Video: `http://becublog.com/math414/#card7` (watch first)

How to perform: This is a self-working trick. First, you must know the card on top of the deck ahead of time. Second, you must find the largest power of 2 less than the spectator's number (call their number k). Then double the difference between the spectator's number and this power of 2, i.e. $2 * (k - 2^i)$. Suppose k=13, then the key number you have to remember is $2 * (13 - 2^3) = 10$. In order for the Australian Shuffle (next step) to work as planned, you must deal so that your 10th card (from top of the deck) is also the 10th card of your new pile (13 cards in this case). The Australian Shuffle makes sure that the top card of the deck will become the 10th card, hence the last card in your hand. If the call out number is a power of 2 then deal them all out into 1 single pile. Collect cards as instructed above, then do the Australian Shuffle.

Math: In an Australian Shuffle the first card goes down on the table and the next card goes to the bottom of the stack. Repeat this process until there is only 1 card left. This is also known as the Down/Under deal.

Claim: The Australian Shuffle keeps exactly one card. In other words, let n be the chosen number, then the position (from the top) of the card which will be kept is 2 * (n - 2^i) where 2^i is the highest power of 2 less than n.

Proof: This proof uses the binary representation of integers.

Suppose n is some power of 2, say n = 2^m. Then after first down-under round, all the odd-numbered cards (i.e. 1, 3, 5, 7th...) are dealt onto the table. We're left with all even numbered cards. For the second down-under round, all the 2x multiples of the first round are dealt onto the table (i.e. 2, 6, 10, 14th...). For the third down-under round, all 2x multiples of the second round are dealt onto the table (i.e. 4, 12, 20, 28th...), or equivalently, all 4x multiples of th first round are dealt onto table. Continuing in the manner, we can easily see that the last card to remain in the magician's hand is the $2^m - th$ card, i.e. the bottom card of the original stack.

Now suppose n is not a power of 2. Suppose that the binary form of n is 1bcd...z where b, c, d, ..., z are either 1 or 0. Then the binary form of highest power of 2 less than n is 1000...0. The difference is bcd...z. When we multiply by 2, it becomes bcd...z0. (1)

We claim that the card in this position will be the last one in the magician's hand. Think of this as creating a sequence of stacks, each one shorter then the previous stack. Each new stack is formed after as many down/under pairs of cards as possible have been dealt. For example: say we have 5 cards from 1 to 5. Then the first stack is 1 2 3 4 5, and the possible down/under pairs are: 1/2, 3/4, and 5/2. Hence the new stack is 4 2. Notice that if the previous stack has $2k+1$ cards, then the new stack will have n cards with its bottom card being the previous stack's second (from top) card. On the other hand, if the previous stack has $2k$ cards, then the new stack will also have n cards but its bottom card is the same as the bottom card in the previous stack. Similar to the reasoning above, when n is some power of 2, each new stack is a 2^kx multiple of the original ordering of the first stack.

In the language of binary, the above statements imply that the number of cards in stack $(n+1)$ is obtained by deleting the last digit of the number of cards in stack n. So if the last digit was 1, the new bottom card has original order 2^n larger than the previous bottom one. So the original order of the last bottom card is the sum of 2^{i+1} for each i such that the number of cards in the original stack has 1 in the i^{th} position, counting from the right starting with 0. The leftmost 1 does not count because we just created a new stack (since we required as many as possible down/under pairs). This is equivalent to (1) above, hence completes the proof. QED.

8 Super Awesome Best Trick Great Job

Goal: Know exactly where in the deck a card is, along with its suit and value, based on the total of values of face up cards dealt on table.

Video: `http://becublog.com/math414/#card8` (watch first)

How to perform: For historical reasons, it should noted that this trick dates back to the mid 1700's in the Caribbean, where pirates would use it to win over women and impress their inebriated friends after a long night of drinking. For this trick, one has a participant take a deck of 52 cards and flip over 25 cards from the top of the deck. As the person places each card down face up into a pile, pretend that you are counting or even memorizing the cards, while in reality you are waiting to observe the 17th card. As the 17th card enters the pile, remember it for your life, for much depends on your performance with this trick (future wealth, marriage, etc.). After 25 cards are in a pile, flip them over and place them to the side. Take the other pile and flip over the top card. If this card is a 10 through Ace, place it at the bottom of the pile and draw the next card from the top. If it is a 2 through 9, place it on the table face up and then begin to count up to 10 from the face value of the card, each time placing a card face down in a pile next to the face up card. For example, if a 3 turns up, place it down face up on the table, and then begin to place cards face down into a pile next to it, counting "4,5,6,...10." In other words, their will be 7 cards face down next to this 3.

Once a face down pile has been created, repeat this procedure 3 more times. When all is said and done, there will be 4 face up cards with 4 face down piles. Take the remaining cards you have and place them on top of the 25 cards your victim counted in the beginning. Then add up the values of the 4 face up cards. This number will be the position of 17th card in the original pile of 25. Since you know what this card is, say something along the lines of, "since these cards add up to X, I am predicting that the X-th card is the blah blah." Say it just like that.

Math: The card that you remember in the beginning is 17th from the top once the deck is flipped over and placed to the side. You are left with 27 cards with which to perform your magic. Let c_i denote the value of each face up card you place down on the table. Then each face down pile will consist of $10 - c_i$ cards. There are 4 face down piles and 4 face up cards. So the total amount of cards removed from the 27 is

$$4 + \sum_{i=1}^{4}(10 - c_i)$$

So we are left with

$$27 - [4 + \sum_{i=1}^{4}(10 - c_i)] = 23 - \sum_{i=1}^{4}(10 - c_i) = \sum_{i=1}^{4} c_i - 17$$

cards. We then take this amount of cards and place it on top of the pile of 25, which changes the position of the 17th card to position

$$17 + \sum_{i=1}^{4} c_i - 17 = \sum_{i=1}^{4} c_i$$

9 The Most Complicated/Least Complicated Math Based Card Trick Ever Invented

Goal: Of course, we save the best for last. Start out with 2 stacks of cards. "Black" deck (spectator deck), b, contains ace to king of spades. "Red" deck (magician deck), r, contains ace to king of hearts. Cards in both decks (face down) are in some order arranged by the magician. Using his deck, the magician will be able to tell the position of any cards in the spectator's deck.

Video: `http://becublog.com/math414/#card9` (watch first)

Setup part 1: First, the face values of Jack, Queen, and King are 11, 12, and 13 respectively, but for now you can place the King of spades and King of hearts to the side. Cards in the black deck are ordered so that the face values of successive cards represent the successive power residues modulo 13 of its primitive root 2. In other word, $b_i \cong 2^{i-1}$ (mod 13). Below is the original ordering of black deck:

i =	1	2	3	4	5	6	7	8	9	10	11	12
$b_i =$	1	2	4	8	3	6	12	11	9	5	10	7

with 1 being the top card of the deck face down. The magician then cuts the black deck arbitrarily to get a new sequence of b_n, say 5 10 7 1 2 4 5 3 6 12 11 9. This is where the demonstration video above starts. To obtain a corresponding red deck, consider the above sequence as a permutation δ which sends elements of the set b_n to the set of integers from 1 to 12. So $5 \to 1$, $10 \to 2$, $7 \to 3$,...The ordering of elements in the set r_n is δ^{-1}, the inverse of δ. The following is the black deck and its corresponding red deck:

$b_n =$	5	10	7	1	2	4	8	3	6	12	11	9
n =	1	2	3	4	5	6	7	8	9	10	11	12
$r_n =$	4	5	8	6	1	9	3	7	12	2	11	10

Table 1: Initial black and red decks

Collect cards face down in order, so that 5 is the top card of black deck and 4 is the top card of red deck.

Perform part 1: At this point, you are ready for the first part of the trick. Two decks setup this way will have what is called a "reciprocity" property. Ask your spectator a card value he wants to find in his deck. Say it's the 7 of spades. All you have to do is count 7 cards from the top card. The face value of your 7th card (which is the 3 of hearts in this case) is the position (from the top) of his/her 7 of spades. This goes both ways, meaning that your spectator can easily find any card in your deck following the same procedure. This gets boring pretty quick, so lets move on to the real deal.

Setup & Performance: First lets learn about "**Red k-shuffle**":

Take your King of hearts and put it underneath your deck. Do some arbitrary cuts to your red deck, for example (b_n is left fixed for now):

$b_n =$	5	10	7	1	2	4	8	3	6	12	11	9	
n =	1	2	3	4	5	6	7	8	9	10	11	12	13
$r_n =$	3	7	12	2	11	10	13	4	5	8	6	1	9

Let the spectator choose a number k between 1 and 12, say k=4. So now your deck has 13 cards while your spectator's deck only has 12 cards. You will then 'shuffle' the red deck using whats called 'Red k-shuffle'. Deal your deck from left to right, and once you reach k cards on each row, start a new row and continue until you're out of cards. Below is an example when k=4.

3	7	12	2
11	10	13	4
5	8	6	1
9			

Notice that we have 4 columns. These are called "heaps". Label them from 1 to 4. Now collect cards in each heap so that 3, 7, 12, and 2 are the bottom cards of each heap. Note that your last card (9 of hearts) falls on heap z = 1. Now, we're going to combine all your heaps as follows: let your spectator choose your starting heap (2 in the video). From there, pick up all the cards in your first heap and then go to the right cyclically, each time z positions. At the end of every z cycle, pick up all cards of your current heap. Repeat this process until you have 1 deck. Finally, cut the deck once to bring the King of hearts to the end of your deck. By doing this Red k-shuffle, we indeed break the special bond (mutual inverse) between the black deck and the red deck, but in fact you only need to do one cut in the black deck to bring that relationship back to life as follows: locate the new position number n of the Ace of hearts, then cut the black deck so that the face value n occupies the first position. If you've been following this, you should get the table of the new sequence of the black deck B_n and red deck R_n below:

$B_n =$	2	4	8	3	6	12	11	9	5	10	7	1	
n =	1	2	3	4	5	6	7	8	9	10	11	12	13
$R_n =$	12	1	4	2	9	5	11	3	8	10	7	6	13

Math: This trick gives rise to some interesting ideas.

Reciprocity: This is obvious by the construction of the order of cards in the black deck and red deck. Since they are mutual inverses of each other, we can find one by knowing the image or preimage of the other card.

Some Basic Formulas: From Table 1, let f = the face value of the first card right after the Ace of spades (cyclically) and e = the number of cards at the end of the black deck after the Ace of spades. x(n) = the index of n mod 13. So here we have f = 2 (primitive root), e = 8. By definition, position i^{th} is occupied by a card with face value

$$b_i \cong f^j \text{ (mod 13) where } j \cong i + e \text{ (mod 12)}. \qquad (1)$$

For the red deck, position n^{th} is occupied by a card of face value

$$r_n = i \cong x(n) - e \text{ (mod 12)} \qquad (2)$$

Red k-shuffle: WLOG, consider a deck of diamonds from 1 to p = 13 which we're going to perform a k-shuffle on. Example below shows when k=5,

1	2	3	4	5
6	7	8	9	10
11	12	13		

As in the How to Perform section, the last card will fall on heap z where z = p - mk ⇒ p = mk + z for some integer m. Since p = 13 is prime, we have gcd(z, p) = 1. Therefore, we are never going to land on an empty heap before all the heaps have been picked up. Consider heap #i, its bottom card has face value i, and its top card has face value $mk + i$ if i ≤ z or $(m - 1)k + i$ if i > z. Notice that within each heap, successive cards have the same difference, namely k. When heap #i is placed on top of heap #i+z (mod k), the difference d between the bottom card of heap# i and top card of heap #i+z is: d =

$$\text{i - [mk + (i + z - k)] = k - p} \cong \text{k (mod p) if i+z} > \text{k}$$

$$\text{i - [(m-1)k + (i + z)] = k - p} \cong \text{k (mod p) if i+z} \leq \text{k}$$

So the process of collecting heaps makes sure that successive cards will have a constant difference in face value, namely k, not only within a heap but between heaps, thus creating a bigger heap with the same property. Hence with a k-shuffle, we always end up with a deck where each card is cyclically k apart from its neighborhood cards. Again since p = 13 is prime, for any choice of k, gcd(k, p) = 1 so we can never land on an empty position, but on a spot occupied by a card until all p cards have been selected.

Remember how we allow the magician to perform arbitrary cuts while the spectator chooses a $k < 13$? That is because arbitrary cuts, and even arbitrary choices of the starting heap (by spectator) won't have any effect because of the cyclic nature of the problem. We've seen that a similar act is done in "Symmetry, take it easy" above.

We can now conclude that a similar result will happen to the deck of hearts of the magician. In other words, the transformation of the sequence $\{r_n\} = r_1, r_2, ..., r_{13}$ into sequence $\{R_n\} = r_k, r_{2k}, ..., r_{13k}$. So $R_n = r_{nk}$

Keeping Reciprocity in line: After the Red k-shuffle, we need to adjust the black deck to maintain reciprocity. From (2), we have

$$R_n = r_{nk} \cong \text{x(nk) - e (mod 12)}$$

From the fundamental property of indices for primes (by Oystein Ore),

$$x_{nk} \cong \text{x(n) + x(k) (mod p-1)}$$

we get

$$R_n \cong r_n + r_k + \text{e} = r_n + (R_1 + \text{e}) \text{ (mod 12)}$$

So, the effect of a k-shuffle is increasing the face value of each card in the red deck by the same constant.

For the black deck, r_i = i corresponds to b_i = n. So if we have the new sequence R_n for the red deck, if R_n = j then B_j = n where j ≅ i + (R_1 + e) (mod 12). This clarifies the fact that we only need to move the $R_1 + e$ cards from the bottom of the black deck to the top. So from (2), the number of black cards to move is:

$$m \cong r_k + e \cong x(k) \pmod{12}$$

which we can see depends only on k. The black deck also needs to be updated by the movement of m cards where $k \cong f^m \pmod{13}$ in the red deck. This concludes the awesome Mathematics explanation for the trick.

Discussion: Part 1 of the trick can be manipulated to make it last a couple more tries. Instead of telling the spectator his/her card will be in exactly the k^{th} position from the top, we can always make him/her start from the bottom.

During the k-shuffle, if the last card happens to fall closer to the right-most heap than the leftmost one, we can proceed toward the left, but one must note that you have to reverse the label of heaps and the first heap must be 0 (not 1). So z' in this case is z-1 and everything else is the same.

Nothing stops us from doing just one k-shuffle. In fact, the magician can perform two, or even three k-shuffles each with different k from different spectators. What's even cooler is that one can alternate the spectator's regular cut and the magician's k-shuffle and it doesn't matter how many times the red deck is being cut and the k-shuffle performed, reciprocity holds eventually.

Kurt Eisemann in his published work offered an even deeper extension of the trick. Instead of applying the k-shuffle to the red deck, one can try to apply the k-shuffle to black deck. What seems like a very similar task suddenly becomes quite complicated but doable. Refer to Kurt Eisemann's published work for more detail.

Of course, there is nothing special about the prime number 13. We only limit to 13 since we're dealing with cards. In reality, everything we said above can be applied to a more general problem for any prime p.

While testing this trick out, creating a corresponding red deck for any given black deck is simple but tedious. We wrote a little code to quickly generate the red deck. We suggest you should do the same. We lacked the sophistication to put the code into LaTex, so contact us if you need any help.

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