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The first few nonmaximal orders attached to weight two newforms on $\Gamma_0(N)$

W.A. Stein
Department of Mathematics, University of California, Berkeley, CA 94720, USA

Introduction

Let $f \in S_2(\Gamma_0(N))$ be a weight two newform. Attach to f the ring

$$\mathcal{O}_f = \mathbf{Z}[\ldots a_n \ldots]$$

generated by the Fourier coefficients of f. This ring is an order in the totally real number field $K_f = \mathbf{Q}[\ldots a_n \ldots]$. David Carleton asked me for examples in which $\mathcal{O}_f$ is not equal to the maximal order in K_f, equivalently, for which $\mathrm{Spec}(\mathcal{O}_f)$ is not normal. Armand Brumer pointed out that Birch had long ago laboriously compiled tables of f so that $\mathcal{O}_f$ appeared to be non-maximal. Unfortunately, we do not have access to Birch's tables and it was perhaps possible that in Birch's examples, given the limited computing resources of the day, that he had not computed enough of the a_n's to generate the full ring $\mathcal{O}_f$. When $N = 1$ and the weight grows, Jochnowitz showed that $\mathcal{O}_f$ is far from normal.

Data gathering

Using `HECKE` [3], the author computed the index of $\mathcal{O}_f$ in the maximal order for every newform f of level $N \leq 450$. This was done as follows:

1. Compute enough a_n to generate $\mathcal{O}_f$ as a $\mathbf{Z}$-module using the Sturm bound [1, 2, 4].
2. Compute the discriminant of $\mathcal{O}_f$ by embedding a matrix representation of $\mathcal{O}_f$ into a space of column vectors, finding an integral basis, and computing the determinant of the trace pairing.
3. Compute the discriminant of K_f using `LiDIA`.

There are 1775 newforms of level $N \leq 450$. Of these, 93 have the property that $\mathcal{O}_f$ is *not* maximal. The distribution is as follows, where index is the index of $\mathcal{O}_f$ in the maximal order.

Table 1. Distribution of indexes

index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
number	1682	77	1	12	1	0	0	1	0	0	0	0	0	0	0	1

Table 2. Each newform of level ≤ 450 with $\mathcal{O}_f$ having given index:

index	f
2	69B, 77D, 105B, 136C, 138D, 154D, 160C, 165C, 171E, 195E, 207D, 213E, 217D, 221D, 221G, 224C, 224D, 225F, 226D, 238F, 255D, 256E, 260B, 272E, 273E, 282E, 299B, 299G, 301D, 305C, 305D, 310E, 315F, 320G, 322E, 322G, 323E, 329F, 337B, 340B, 355E, 357H, 359D, 363G, 368I, 376D, 377F, 385H, 390H, 392G, 399F, 399G, 406G, 410E, 410G, 410I, 414F, 415D, 416D, 417F, 426G, 429H, 433C, 434I, 435E, 435J, 437F, 438H, 438I, 442F, 442H, 442I, 445A, 445F, 445G, 448I, 448J
3	271B
4	219E, 291H, 293B, 303E, 371E, 387J, 389E, 395H, 413F, 416F, 431F, 437H
5	401B
8	371F
16	257B

Table 3. The first example exhibiting each index.

index	f	$[K_f : \mathbf{Q}]$	$\mathrm{disc}(K_f)$
2	69B	2	5
3	271B	16	$1367 \cdot 6091 \cdot 1132673 \cdot 14171513 \cdot 172450541$
4	219E	6	$2^2 \cdot 1189637$
5	401B	21	$2^8 \cdot 19 \cdot 163 \cdot 71742740351 \cdot 388881803749 \cdot 34393898968391$
8	371F	11	$2^8 \cdot 157 \cdot 76723322773093$
16	257B	14	$2^7 \cdot 29 \cdot 479 \cdot 71711 \cdot 409177 \cdot 654233$

References

[1] A. Agashe and W.A. Stein, *Generating the Hecke algebra*, Preprint (1999).

[2] W.A. Stein, *Congruences between modular forms*, Notes (1999).

[3] ———, `HECKE`*: The modular forms calculator*, Software (available online) (1999).

[4] J. Sturm, *On the congruence of modular forms*, Number theory (New York, 1984–1985), Springer, Berlin, 1987, pp. 275–280.