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Component groups of optimal quotients of Jacobians

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Introduction

Let A be an abelian variety over a finite extension K of the p-adic numbers $\mathbf{Q}_p$. Let $\mathcal{O}$ be the ring of integers of K, $\mathfrak{m}$ its maximal ideal and $k = \mathcal{O}/\mathfrak{m}$ the residue class field. The Néron model of A is a smooth commutative group scheme $\mathcal{A}$ over $\mathcal{O}$ such that A is its generic fiber and satisfying the property:
the restriction map

$$\mathrm{Hom}_{\mathcal{O}}(S, \mathcal{A}) \longrightarrow \mathrm{Hom}_K(S/K, A)$$

is bijective for all schemes S over $\mathcal{O}$. The special fiber $\mathcal{A}_k$ is a group scheme over k, which need not be connected. Denote by $\mathcal{A}_k^0$ the connected component containing the identity. There is an exact sequence

$$0 \longrightarrow \mathcal{A}_k^0 \longrightarrow \mathcal{A}_k \longrightarrow \Phi_A \longrightarrow 0$$

with Φ_A a finite étale group scheme over k, i.e., a finite abelian group equipped with an action of $\mathrm{Gal}(\overline{k}/k)$.

In this paper we study the group Φ_A with particular emphasis on quotients A of Jacobians of modular curves $X_0(N)$. When A has semistable reduction Grothendieck described the component group in terms of a monodromy pairing on certain free abelian groups. When $A = J$ is the Jacobian of $X_0(N)$, this pairing can be explicitly computed, hence so can Φ_J, as has been done in many cases in [Maz77] and [Edi91]. Suppose now that A is a simple quotient of J and that the kernel of the map $J \to A$ is connected. There is a natural map $\Phi_J \to \Phi_A$. In this paper we give a formula which can be used to compute the image and the order of the cokernel.

We now state our main result in more precise language. Suppose $\pi : J \to A$ is an optimal quotient, with J a semistable Jacobian and A purely toric. We express the component group of A in terms of the monodromy pairing associated to J. Let $m_A = \sqrt{\deg(\theta_A)}$ where $\theta_A : A' \to A$ is induced by the canonical principal polarization of J. Let X_J be the character group of the toric part of the special fiber of J. Let $\mathcal{L}$ be the saturation of the image of X_A in X_J. The monodromy pairing defines a map $\alpha : X_J \to \mathrm{Hom}(\mathcal{L}, \mathbf{Z})$. Let Φ_X be the cokernel of α and $m_X = [\alpha(X_J) : \alpha(\mathcal{L})]$ be the order of the finite group $\alpha(X_J)/\alpha(\mathcal{L})$. We prove that

$$\frac{|\Phi_A|}{m_A} = \frac{|\Phi_X|}{m_X}.$$

More precisely, Φ_X is the image of $\Phi_J \to \Phi_A$ and the cokernel of $\Phi_J \to \Phi_A$ has order m_A/m_X. If the optimal quotient $J \to A$ arises from a modular form on $\Gamma_0(N)$, then the quantities m_A, m_X and Φ_X can be explicitly computed, hence so can $|\Phi_A|$. Having done this, we present some tables and conjectures which they suggest.

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1 Optimal quotients of jacobians

Let J be a Jacobian equipped with its canonical principal polarization θ_J. An **optimal quotient** of J is an abelian variety A and a surjective map $\pi : J \to A$ whose kernel is an abelian subvariety B of J. Denote by J' and A' the abelian varieties dual to J and A, respectively. Dualizing π and composing with $\theta'_J = \theta_J$ we obtain a map $A' \xrightarrow{\pi'} J' \xrightarrow{\theta_J} J$.

Proposition 1.1. *$A' \to J$ is injective.*

Proof. Since θ_J is an isomorphism it suffices to prove that π' is injective. Since the dual of π' is $(\pi')' = \pi$ and π is surjective, π' must have finite kernel. Thus $A' \to C = \mathrm{im}(\pi')$ is an isogeny. Let G denote the kernel, and dualize. By [Mil86, §11] we have have

$$\begin{tikzcd} G \arrow[r] & A' \arrow[r, two heads] \arrow[dr, "\pi'"'] & C \arrow[d] \\ & & J' \end{tikzcd} \qquad \xrightarrow{\text{dualize}} \qquad \begin{tikzcd} A & C \arrow[l] & G' \arrow[l] \\ & J \arrow[ul, "\pi"] \arrow[u, "\varphi"'] & \end{tikzcd}$$

with G' the Cartier dual of G. Since G' is finite, $\ker(\varphi) \subset \ker(\pi)$ is of finite index. Since $\ker(\pi)$ is an abelian variety it is divisible. Thus $\ker(\varphi) = \ker(\pi)$ so $G' = 0$ and hence $G = 0$. $\square$

We denote the map $A' \to J$ by π'. The kernel of θ_A measures the intersection of A' and $B = \ker(\theta_A)$ inside of J as shown in the following diagram.

$$\begin{tikzcd}
A' \cap B \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r, hook, "\pi'"] \arrow[dr, "\theta_A"'] & J \arrow[d, "\pi"] \\
& A
\end{tikzcd}$$

Because θ_A is a polarization, $|\ker(\theta_A)|$ is a square [Mil86, Theorem 13.3]. The **congruence modulus** is the integer

$$m_A = \sqrt{|\ker(\theta_A)|}.$$

2 The special fiber of the Néron model

Let K be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}$ and residue class field k. Let A/K be an abelian variety and denote its Néron model by $\mathcal{A}$. Let Φ_A be the group of connected components of the special fiber $\mathcal{A}_k$. This group is a finite étale group scheme over k, i.e., a finite abelian group equipped with an action of $\mathrm{Gal}(\overline{k}/k)$. There is an exact sequence of group schemes

$$0 \to \mathcal{A}_k^0 \to \mathcal{A}_k \to \Phi_A \to 0.$$

The group $\mathcal{A}_k^0$ is an extension of an abelian variety $\mathcal{B}$ of dimension a by the product of a torus $\mathcal{T}$ of dimension t and of a unipotent group $\mathcal{U}$ of dimension u:

$$0 \to \mathcal{U} \times \mathcal{T} \to \mathcal{A}_k^0 \to \mathcal{B} \to 0.$$

The abelian variety A is said to have **purely toric** reduction if $t = \dim A$, and is **semistable** if $u = 0$. The character group $X_A = \mathrm{Hom}(\mathcal{T}, \mathbf{G}_m)$ is a free abelian group of rank t contravariantly associated to A. If A is semistable there is a monodromy pairing $X_A \times X_{A'} \to \mathbf{Z}$ and an exact sequence

$$0 \to X_{A'} \to \mathrm{Hom}(X_A, \mathbf{Z}) \to \Phi_A \to 0.$$

3 Rigid uniformization

In this section we review the rigid analytic uniformization of a semistable abelian variety over a finite extension K of the maximal unramified extension $\mathbf{Q}_p^{\mathrm{ur}}$ of $\mathbf{Q}_p$. We use this to prove that if A is purely toric, and $\phi : A' \to A$ is an isogeny, then

$$\deg(\phi) = |\operatorname{coker}(X_A \to X_{A'})|^2.$$

We also prove a few lemmas about the character groups X_A.

3.1 Raynaud-van der Put uniformization

Theorem 3.1 (Raynaud, van der Put). *If A is a semistable Abelian variety, its universal covering is isomorphic to an extension G of an abelian variety B with good reduction by a torus T, the covering map from G to A is a homomorphism and its kernel is a twisted free Abelian group Γ of finite rank.*

This may be summarized by the following diagram,

$$\begin{tikzcd} & \Gamma \arrow[d] & \\ T \arrow[r] & G \arrow[r] \arrow[d] & B \\ & A & \end{tikzcd}$$

which we call the **uniformization cross** of A. The group Γ can be identified with the character group X_A of the previous section. The uniformization cross of the dual abelian variety A' is

$$\begin{tikzcd} & \Gamma' \arrow[d] & \\ T' \arrow[r] & G' \arrow[r] \arrow[d] & B' \\ & A' & \end{tikzcd}$$

where $\Gamma' = \mathrm{Hom}(T, \mathbf{G}_m)$ and $T' = \mathrm{Hom}(\Gamma, \mathbf{G}_m)$ and the morphisms $\Gamma' \to G'$ and $T' \to G'$ are the one-motive duals of the morphisms $T \to G$ and $\Gamma \to G$, respectively. For more details see [Col99].

Example 3.2 (Tate curve). If $E/\mathbf{Q}_p$ is an elliptic curve with multiplicative reduction then the uniformization is $E = \mathbf{G}_m/q^{\mathbf{Z}}$ where $q = q(j)$ is obtained by inverting the expression for j as a function of $q(z) = e^{2\pi i z}$.

3.2 Some lemmas

Let $\pi : J \to A$ be an optimal quotient, with J semistable and A purely toric.

Lemma 3.3. *The map $\Gamma_J \to \Gamma_A$ induced by π is surjective.*

Proof. Because G_J is simply connected, π induces a map $G_J \to T_A$ and a map $\Gamma_J \to \Gamma_A$. Because π is surjective and $T(A)$ is a torus, the map $G_J \to T_A$ is surjective. The snake lemma applied to the following diagram gives a surjective map from $B = \ker(\pi)$ to $M = \operatorname{coker}(\Gamma_J \to \Gamma_A)$.

$$\begin{tikzcd}
 & \Gamma_J \arrow[r] \arrow[d] & \Gamma_A \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & G_J \arrow[r] \arrow[d] & T_A \arrow[r] \arrow[d] & 0 & \\
B \arrow[r] & J \arrow[r, "\pi"] & A & &
\end{tikzcd}$$

Because π is optimal, B is connected so M must also be connected. Since M is discrete it follows that $M = 0$. $\square$

3.3 Purely toric abelian varieties

Assume that A is purely toric. Then $B = 0$, and the uniformization cross becomes

$$\begin{tikzcd}
\Gamma \arrow[d] \\
T \arrow[d] \\
A
\end{tikzcd}$$

Let $\varphi : A' \to A$ be a **symmetric isogeny**, i.e., $\varphi' : A' \to (A')' = A$ is equal to φ. Denote by $\varphi_t : T' \to T$ and $\varphi_a : \Gamma' \to \Gamma$ the induced maps.

Proposition 3.4. *There is an exact sequence*

$$0 \to \ker(\varphi_t) \to \ker(\varphi) \to \operatorname{coker}(\varphi_a) \to 0,$$

and $\ker(\varphi_t)$ *is dual to* $\operatorname{coker}(\varphi_a)$.

Proof. Since φ is an isogeny we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Gamma' \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \operatorname{coker}(\varphi_a) \arrow[d] \\
\ker(\varphi_t) \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
\ker(\varphi) \arrow[r] & A' \arrow[r, "\varphi"] & A &
\end{tikzcd}$$

The snake lemma then gives the claimed exact sequence. For the second assertion observe that the one-motive dual of the diagram

$$\begin{tikzcd} & \Gamma' \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \mathrm{coker}(\varphi_a) \\ \ker(\varphi_t) \arrow[r] & T' \arrow[r, "\varphi"] & T & \end{tikzcd}$$

is the diagram

$$\begin{tikzcd} & T & T' \arrow[l, "\varphi'"'] & \mathrm{coker}(\varphi_a)' \arrow[l] \\ \ker(\varphi_t)' & \Gamma \arrow[l] \arrow[u] & \Gamma' \arrow[l] \arrow[u] & \end{tikzcd}$$

Since φ is symmetric, $\varphi' = \varphi$ and so

$$\ker(\varphi_t) = \mathrm{coker}(\varphi_a)'.$$

□

Lemma 3.5. $|\ker(\varphi)| = |\,\mathrm{coker}(\varphi_a)|^2$

Proof. The order of a finite group scheme equals the order of its dual. □

4 The main theorem

Let $\pi : J \to A$ be an optimal quotient, with J a semistable Jacobian and A purely toric. Let X_A, $X_{A'}$, and X_J denote the character groups of the toric parts of the special fibers.

4.1 Monodromy description of the component group

There is a pairing $X_A \times X_{A'} \to \mathbf{Z}$ called the monodromy pairing. We have an exact sequence

$$0 \to X_{A'} \to \mathrm{Hom}(X_A, \mathbf{Z}) \to \Phi_A \to 0.$$

If J is a Jacobian then J is canonically self-dual so the monodromy pairing on J can be viewed as a pairing $X_J \times X_J \to \mathbf{Z}$ and there is an exact sequence

$$0 \to X_J \to \mathrm{Hom}(X_J, \mathbf{Z}) \to \Phi_J \to 0.$$

Example 4.1 (Tate curve). Suppose $E = \mathbf{G}_m/q^{\mathbf{Z}}$ is a Tate curve. The monodromy pairing on $X_E = q^{\mathbf{Z}}$ is

$$\langle q, q\rangle = \mathrm{ord}_p(q) = -\,\mathrm{ord}_p(j).$$

Thus Φ_E is cyclic of order $-\,\mathrm{ord}_p(j)$.

4.2 Proof of the main theorem

We now prove the main theorem. The key diagrams are

$$
\begin{tikzcd}
A' \arrow[r, hook, "\pi'"] \arrow[dr, "\theta"'] & J \arrow[d, two heads, "\pi"] \\
& A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_A \arrow[r, hook, "\pi^*"] \arrow[dr, "\theta^*"] & X_J \arrow[d, two heads, "\pi_*"] \\
& X_{A'} \arrow[ul, bend left, "\theta_*"]
\end{tikzcd}
$$

The surjectivity of π_* was proved in Lemma 3.3. The injectivity of π^* follows because

$$\theta_* \pi_* \pi^* = \theta_* \theta^* = \deg(\theta) \neq 0,$$

and multiplication by $\deg(\theta)$ on a free abelian group is injective.

Let

$$\alpha : X_J \to \mathrm{Hom}(\pi^* X_A, \mathbf{Z})$$

be the map defined by the monodromy pairing restricted to $X_J \times \pi^* X_A$.

Lemma 4.2. $\ker(\pi_*) = \ker(\alpha)$

Proof. Suppose $x \in \ker(\pi_*)$ and let $y = \pi^* z$ with $z \in X_A$. Then

$$\langle x, y \rangle = \langle x, \pi^* z \rangle = \langle \pi_* x, z \rangle = 0$$

so $x \in \ker(\alpha)$. Next let $x \in \ker(\alpha)$. Then for all $z \in X_A$,

$$0 = \langle x, \pi^* z \rangle = \langle \pi_* x, z \rangle$$

so $\pi_* x$ is in the kernel of the monodromy map

$$X_{A'} \to \mathrm{Hom}(X_A, \mathbf{Z}).$$

Since $X_{A'}$ and $\mathrm{Hom}(X_A, \mathbf{Z})$ are free of the same rank and the cokernel is torsion, the monodromy map is injective. Thus $\pi_* x = 0$ and $x \in \ker(\pi_*)$. $\square$

Lemma 4.3. *There is an exact sequence*

$$X_J \to \mathrm{Hom}(\pi^* X_A, \mathbf{Z}) \to \Phi_A \to 0.$$

Proof. Lemma 4.2 gives the following commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & X_J / \ker(\alpha) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(\pi^* X_A, \mathbf{Z}) \arrow[r] \arrow[d, "\cong"] & \mathrm{coker}(\alpha) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_{A'} \arrow[r] & \mathrm{Hom}(X_A, \mathbf{Z}) \arrow[r] & \Phi_A \arrow[r] & 0
\end{tikzcd}
$$

By Lemma 4.2, the first vertical map is an isomorphism. The second is an isomorphism because it is induced by the isomorphism $\pi^* : X_A \to \pi^* X_A$. It follows that $\mathrm{coker}(\alpha) \cong \Phi_A$, as claimed. $\square$

Let $\mathcal{L}$ be the **saturation** of $\pi^* X_A$ in X_J, i.e., $[\mathcal{L} : \pi^* X_A]$ is finite and $X_J/\mathcal{L}$ is torsion free. Suppose L is of finite index in $\mathcal{L}$. Define the **congruence modulus** of L

$$m_L = [\alpha(X_J) : \alpha(L)]$$

and the **component group** by

$$\Phi_L = \mathrm{coker}(X_J \to \mathrm{Hom}(L, \mathbf{Z})).$$

When $L = \mathcal{L}$ we often set $m_X = m_{\mathcal{L}}$ and $\Phi_X = \Phi_{\mathcal{L}}$ and think of m_X and Φ_X as the character group "congruence modulus and component group."

Lemma 4.4. *The rational number* $\dfrac{|\Phi_L|}{m_L}$ *does not depend on the choice of* L.

Proof. If L' is another choice let $n = [L : L'] \in \mathbf{Q}$. Then since α is injective when restricted to $\mathcal{L}$,

$$m_{L'} = [\alpha(X_J) : \alpha(L')] = [\alpha(X_J) : \alpha(L)] \cdot [\alpha(L) : \alpha(L')] = m_L \cdot n$$

and similarly $|\Phi_{L'}| = |\Phi_L| \cdot n$. □

Recall that we defined

$$\begin{aligned} m_A &= \sqrt{\deg(\theta)} \\ \Phi_A &= \mathrm{coker}(X_{A'} \to \mathrm{Hom}(X_A, \mathbf{Z})) \end{aligned}$$

Theorem 4.5. *For any* L *of finite index in* $\mathcal{L}$ *the following relation holds:*

$$\frac{|\Phi_A|}{m_A} = \frac{|\Phi_L|}{m_L}.$$

Proof. By Lemma 4.4 we may assume that $L = \pi^* X_A$. With this choice of L, Lemma 4.3 says that $\Phi_L \cong \Phi_A$. By Lemma 4.2, properties of the index, and Lemma 3.5 we have

$$\begin{aligned} m_L &= [\alpha(X_J) : \alpha(L)] \\ &= [\pi_*(X_J) : \pi_*(L)] \\ &= [X_{A'} : \pi_*(\pi^* X_A)] \\ &= [X_{A'} : \theta^* X_A] \\ &= \mathrm{coker}(\theta^*) \\ &= \sqrt{\deg(\theta)} = m_A. \end{aligned}$$

□

Proposition 4.6.

$$\mathrm{image}(\Phi_J \to \Phi_A) \cong \Phi_{\mathcal{L}}.$$

Proof. Because $\pi^* X_A \subset \mathcal{L} \subset X_J$, by Lemma 4.3 we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
X_J \arrow[r] \arrow[d, equal] & \mathrm{Hom}(X_J, \mathbf{Z}) \arrow[r] \arrow[d] & \Phi_J \arrow[r] \arrow[d] & 0 \\
X_J \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\mathcal{L}, \mathbf{Z}) \arrow[r] \arrow[d] & \Phi_{\mathcal{L}} \arrow[r] \arrow[d] & 0 \\
X_J \arrow[r] & \mathrm{Hom}(\pi^* X_A, \mathbf{Z}) \arrow[r] & \Phi_A \arrow[r] & 0
\end{tikzcd}$$

The map $\mathrm{Hom}(\mathcal{L}, \mathbf{Z}) \to \mathrm{Hom}(\pi^* X_A, \mathbf{Z})$ is an isomorphism so $\Phi_{\mathcal{L}} \to \Phi_A$ is injective, hence

$$\mathrm{image}(\Phi_J \to \Phi_A) \cong \mathrm{image}(\Phi_J \to \Phi_{\mathcal{L}}).$$

The cokernel of $\mathrm{Hom}(X_J, \mathbf{Z}) \to \mathrm{Hom}(\mathcal{L}, \mathbf{Z})$ surjects onto the cokernel of $\Phi_J \to \Phi_{\mathcal{L}}$. Using the exact sequence

$$0 \to \mathcal{L} \to X_J \to X_J/\mathcal{L} \to 0,$$

we find that

$$\mathrm{coker}(\mathrm{Hom}(X_J, \mathbf{Z}) \to \mathrm{Hom}(\mathcal{L}, \mathbf{Z})) \subset \mathrm{Ext}^1(X_J/\mathcal{L}, \mathbf{Z}) = 0,$$

where Ext^1 vanishes because $\mathcal{L}$ is saturated so that $X_J/\mathcal{L}$ is torsion free. Thus the cokernel of $\Phi_J \to \Phi_{\mathcal{L}}$ is 0, from which the proposition follows. □

The following corollary follows from Theorem 4.5 and Proposition 4.6.

Corollary 4.7.

$$|\,\mathrm{coker}(X_J \to X_A)| = \frac{m_A}{m_{\mathcal{L}}}.$$

As a consequence, $m_{\mathcal{L}} | m_A$.

5 Optimal quotients of $J_0(N)$

Let $X_0(N)/\mathbf{Q}$ be the modular curve associated to the congruence subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbf{Z})$ of matrices which are upper triangular modulo N. Let p be a prime divisor of N which is coprime to $M = N/p$. The Jacobian $J = J_0(N)$ of $X_0(N)$ has semistable reduction at p. The Hecke algebra

$$\mathbf{T} = \mathbf{Z}[\ldots T_n \ldots] \subset \mathrm{End}(J)$$

is a commutative ring of endomorphisms of J of $\mathbf{Z}$-rank $= \dim J$. The character group X_J is equipped with a functorial action of $\mathbf{T}$. The Hecke algebra $\mathbf{T}$ also act on the cusp forms

$$S = S_2(\Gamma_0(N), \mathbf{C}).$$

A newform f is an eigenform normalized so that the coefficient of q in the expansion of f at the cusp ∞ is 1, and such that f does not occur at any level $N' \mid N$ with $N' \neq N$. If f is a newform, let I_f be the ideal in $\mathbf{T}$ of elements which annihilate f. Then $\mathcal{O}_f = \mathbf{T}/I_f$ is an order in the ring of integers of the totally real number field K_f obtained by adjoining the Fourier coefficients of f to $\mathbf{Q}$. The quotient

$$A_f = J_0(N)/I_f J_0(N)$$

is a purely toric optimal quotient of dimension $[K_f : \mathbf{Q}]$.

Let $H = H_1(X_0(N), \mathbf{Z})$ be the integral homology of the complex algebraic curve $X_0(N)$. Integration defines a $\mathbf{T}$-equivariant nondegenerate pairing $S \times H \to \mathbf{C}$ which we view as a map $\alpha : H \to \mathrm{Hom}_{\mathbf{C}}(S, \mathbf{C})$.

Theorem 5.1. *We have the following commutative diagram of* $\mathbf{T}$*-modules:*

$$\begin{tikzcd}
H[I_f] \arrow[r, hook] \arrow[d, hook] & H \arrow[r, two heads] \arrow[d, hook] & \alpha(H) \arrow[d, hook] \\
\mathrm{Hom}_{\mathbf{C}}(S, \mathbf{C})[I_f] \arrow[r, hook] \arrow[d, two heads] & \mathrm{Hom}_{\mathbf{C}}(S, \mathbf{C}) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Hom}_{\mathbf{C}}(S[I_f], \mathbf{C}) \arrow[d, two heads] \\
A'_f(\mathbf{C}) \arrow[r, hook] \arrow[rr, bend right, "\theta_A"'] & J(\mathbf{C}) \arrow[r, two heads] & A_f(\mathbf{C})
\end{tikzcd}$$

Proof. This can be deduced from [Shi73]. □

Corollary 5.2. $m_A^2 = [\alpha(H) : \alpha(H[I_f])]$.

Proof. Recall that m_A is defined to be $\sqrt{\deg(\theta_A)}$. The kernel of an isogeny of complex tori is isomorphic to the cokernel of the induced map on lattices. The corollary now follows from the diagram of Theorem 5.1 which indicates that the index $[\alpha(H) : \alpha(H[I_f])]$ is the cokernel of the map $H[I_f] \to \alpha(H)$. □

Let $\mathrm{Frob}_p : X_J \to X_J$ denote the map induced by Frobenius. One has $\mathrm{Frob}_p = -W_p$, where W_p is the map induced by the Atkin-Lehner involution on $J_0(p)$. Let f be a newform, $A = A_f$ the corresponding optimal quotient, and w_p the sign of the eigenvalue of W_p on f.

Proposition 5.3.

$$\Phi_A(\mathbf{F}_p) = \begin{cases} \Phi_A(\overline{\mathbf{F}}_p) & \text{if } w_p = -1, \\ \Phi_A(\overline{\mathbf{F}}_p)[2] & \text{if } w_p = 1. \end{cases}$$

Proof. If $w_p = -1$, then $\mathrm{Frob}_p = 1$ and the $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$-action of $\Phi_A(\overline{\mathbf{F}}_p)$ is trivial. Thus in this case, $\Phi(\mathbf{F}_p) = \Phi(\overline{\mathbf{F}}_p)$. Next suppose $w_p = 1$. We have an exact sequence

$$0 \to X_{A'} \to \mathrm{Hom}(X_A, \mathbf{Z}) \to \Phi_A \to 0.$$

Since W_p acts as $+1$ on f it acts as $+1$ on A, on X_A, on $\mathrm{Hom}(X_A, \mathbf{Z})$, and on Φ_A. Thus $\mathrm{Frob}_p = -W_p$ acts as -1 on Φ_A. The 2-torsion in a finite abelian group equals the fixed points under -1. $\square$

5.1 Computation

Suitable generalizations of the algorithms described in [Cre97] can be used to enumerate the optimal quotients A_f and to compute m_A. These will be described in the author's Berkeley Ph.D. thesis [Ste99a]. The method of graphs [Mes86] and [Koh98] can be used to compute $X = X_{J_0(N)}$ with its $\mathbf{T}$-action and the monodromy pairing. We can then compute

$$\mathcal{L} = \bigcap_{t \in I_f} \ker(t|_X),$$

$m_X := m_{\mathcal{L}}$, and $\Phi_X := \Phi_{\mathcal{L}}$. By Theorem 4.5 we can now compute

$$|\Phi_A| = |\Phi_X| \cdot \frac{m_A}{m_X}.$$

We have computed Φ_A in a number of cases. In the next subsection we discuss two conjectures suggested by our numerical computations.

5.2 Conjectures

Our numerical computations suggest the following conjectures. Suppose that $N = pM$ with $(p, M) = 1$. Let

$$H_{\mathrm{new}} = \ker\Big(H_1(X_0(N), \mathbf{Z}) \longrightarrow (H_1(X_0(M), \mathbf{Z}) \oplus H_1(X_0(M), \mathbf{Z})\Big),$$

where the map is induced by the two natural degeneracy maps $X_0(N) \to X_0(M)$. The Hecke algebra $\mathbf{T}$ acts on H_{new}, and on the submodule $H_{\mathrm{new}}[I_f]$ of elements annihilated by I_f. Integration defines a map $\alpha : H_{\mathrm{new}} \to \mathrm{Hom}(S[I_f], \mathbf{C})$. Define the homology congruence modulus m_H by

$$m_H^2 = [\alpha(H_{\mathrm{new}}) : \alpha(H_{\mathrm{new}}[I_f])].$$

We expect that there is a very close relationship between m_X and m_H.

Conjecture 5.4. *Up to powers of* 2,

$$m_X = m_H.$$

When $N = p$ is prime we make the following conjecture.

Conjecture 5.5. *Let p be a prime and let $f_1, \ldots, f_n$ be a set of representatives of the Galois conjugacy classes for newforms in $S_2(\Gamma_0(p))$. Let $A_1, \ldots, A_n$ be* *the corresponding optimal quotients. Then the natural maps*

$$\begin{array}{rcl} \Phi_{J_0(p)} & \longrightarrow & \prod_{i=1}^{n} \Phi_{A_i} \\ J_0(p)(\mathbf{Q}) & \longrightarrow & \prod_{i=1}^{n} A_i(\mathbf{Q}) \end{array}$$

are isomorphisms.

We offer the tables in the next section as evidence that an assertion such as the above two conjectures may be true.

6 Tables

We computed several component groups of optimal quotients A_f of $J_0(N)$ associated to newforms f. We denote such an optimal quotient by

N isogeny-class dimension

The dimension frequently determines the factor, so it is included in the notation.

6.1 Table 1: Some large component groups predicted by the Birch and Swinnerton-Dyer conjecture

Using the algorithm described in [Ste99b] we computed the special value $L(A,1)/\Omega$ (up to a Manin constant) for every optimal quotient $A = A_f$ of level ≤ 1500. We found exactly five for which the numerator of $L(A,1)/\Omega$ is nonzero and divisible by a prime number $> 10^9$. These are given below.

A	N	$L(A,1)/\Omega \cdot$ Manin constant
1154E20	$2 \cdot 577$	$2^? \cdot 85495047371/17^2$
1238G19	$2 \cdot 619$	$2^? \cdot 7553329019/5 \cdot 31$
1322E21	$2 \cdot 661$	$2^? \cdot 57851840099/331$
1382D20	$2 \cdot 691$	$2^? \cdot 37 \cdot 1864449649/173$
1478J20	$2 \cdot 739$	$2^? \cdot 7 \cdot 29 \cdot 1183045463/5 \cdot 37$

The Birch and Swinnerton-Dyer conjecture predicts that these large prime divisors must divide either $|\Phi_A|$ or the Shafarevich-Tate group of A. We computed Φ_A and found that this was the case.

A	p	w	$\lvert\Phi_X\rvert$	m_X	$\lvert\Phi_A\rvert$
1154E20	2	$-$	17^2	2^{24}	$2^? \cdot 17^2 \cdot 85495047371$
	577	$+$	1	$2^{26} \cdot 85495047371$	$2^?$
1238G19	2	$-$	$5 \cdot 31$	2^{26}	$2^? \cdot 5 \cdot 31 \cdot 7553329019$
	619	$+$	1	$2^{28} \cdot 7553329019$	$2^?$
1322E21	2	$-$	331	2^{28}	$2^? \cdot 331 \cdot 57851840099$
	661	$+$	1	$2^{32} \cdot 57851840099$	$2^?$
1382D20	2	$-$	173	2^{29}	$2^? \cdot 37 \cdot 173 \cdot 1864449649$
	691	$+$	1	$2^{31} \cdot 37 \cdot 1864449649$	$2^?$
1478J20	2	$-$	$5 \cdot 37$	2^{31}	$2^? \cdot 5 \cdot 7 \cdot 29 \cdot 37 \cdot 1183045463$
	739	$+$	1	$2^{33} \cdot 7 \cdot 29 \cdot 1183045463$	$2^?$

6.2 Table 3: Some quotients of $J_0(N)$

In this table we give the invariants defined above for the optimal quotients of levels 65, 66, 68, and 69.

A	p	w	$\lvert\Phi_X\rvert$	m_X	m_H	m_A	$\lvert\Phi_A\rvert$
65A1	5	$+$	1	2	?	2	1
	13	$+$	1	2	?		1
65B2	5	$+$	3	2^2	?	2^2	3
	13	$-$	3	2^2	?		3
65C2	5	$-$	7	2^2	?	2^2	7
	13	$+$	1	2^2	?		1
66A1	2	$+$	1	2	?	2^2	2
	3	$-$	3	2^2	?		3
	11	$+$	1	2^2	?		1
66B1	2	$-$	2	2	?	2^2	2^2
	3	$+$	1	2^2	?		1
	11	$+$	1	2^2	?		1
66C1	2	$-$	1	2	?	$2^2 \cdot 5$	$2 \cdot 5$
	3	$-$	1	2^2	?		5
	11	$-$	1	$2^2 \cdot 5$	?		1
68A2	17	$+$	2	$2 \cdot 3$	?	$2 \cdot 3$	2
69A1	3	$-$	2	2	?	2	2
	23	$+$	1	2	?		1
69B2	3	$+$	2	2	?	$2 \cdot 11$	$2 \cdot 11$
	23	$-$	2	$2 \cdot 11$	?		2

6.3 Table 3: Some quotients of $J_0(p)$

Using the method of graphs and modular symbols we computed the quantities m_A, m_L and Φ_L for each abelian variety $A = A_f$ associated to a newform of prime level $p \leq 757$. The results were as follows:

1. In all cases $m_A = m_L$, so the map $\Phi_J \to \Phi_A$ is surjective.
2. $\Phi_A = 1$ whenever the sign of the Atkin-Lehner involution w_p on A is 1.
3. $\prod |\Phi_A| = |\Phi_J|$

Table 1 lists those A of level ≤ 631 for which $w_p = -1$, along with the order of the component group.

Table 1: Some quotients of $J_0(p)$

A	$\lvert\Phi_A\rvert$	A	$\lvert\Phi_A\rvert$	A	$\lvert\Phi_A\rvert$	A	$\lvert\Phi_A\rvert$
11A1	5	157B7	13	313A2	1	487B2	3
17A1	2^2	163C7	3^3	313C12	$2 \cdot 13$	487C3	1
19A1	3	167B12	83	317B15	79	487D16	3^3
23A2	11	173B10	43	331D16	$5 \cdot 11$	491C29	$5 \cdot 7^2$
29A2	7	179A1	1	337B15	$2^2 \cdot 7$	499C23	83
31A2	5	179C11	89	347D19	173	503B1	1
37B1	3	181B9	$3 \cdot 5$	349B17	29	503C1	1
41A3	$2 \cdot 5$	191B14	$5 \cdot 19$	353A1	2	503D3	1
43B2	7	193C8	2^4	353B3	2	503F26	251
47A4	23	197C10	7^2	353D14	$2 \cdot 11$	509B28	127
53B3	13	199A2	1	359D24	179	521B29	$2 \cdot 5 \cdot 13$
59A5	29	199C10	$3 \cdot 11$	367B19	61	523C26	$3 \cdot 29$
61B3	5	211A2	5	373C17	31	541B24	$3^2 \cdot 5$
67A1	1	211D9	7	379B18	$3^2 \cdot 7$	547C25	$7 \cdot 13$
67C2	11	223C12	37	383C24	191	557B1	1
71A3	5	227B2	1	389A1	1	557D26	139
71B3	7	227C2	1	389E20	97	563A1	1
73A1	2	227E10	113	397B2	1	563E31	281
73C2	3	229C11	19	397C5	11	569B31	$2 \cdot 71$
79B5	13	233A1	2	397D10	3	571A1	1
83B6	41	233C11	29	401B21	$2^2 \cdot 5^2$	571B1	1
89B1	2	239B17	$7 \cdot 17$	409B20	$2 \cdot 17$	571C2	1
89C5	11	241B12	$2^2 \cdot 5$	419B26	$11 \cdot 19$	571D2	1
97B4	2^3	251B17	5^3	421B19	$5 \cdot 7$	571F4	1
101B7	5^2	257B14	2^6	431B1	1	571I18	$5 \cdot 19$
103B6	17	263B17	131	431D3	1	577A2	3
107B7	53	269C16	67	431F24	$5 \cdot 43$	577B2	1
109A1	1	271B16	$3^2 \cdot 5$	433A1	1	577C3	1
109C4	3^2	277B3	1	433B3	1	577D18	2^4
113A1	2	277D9	23	433D16	$2^2 \cdot 3^2$	587C31	293
113B2	2	281B16	$2 \cdot 5 \cdot 7$	439C25	73	593B1	2
113D3	7	283B14	47	443C1	1	593C2	1
127B7	$3 \cdot 7$	293B16	73	443E22	$13 \cdot 17$	593E27	$2 \cdot 37$
131B10	$5 \cdot 13$	307A1	1	449B23	$2^4 \cdot 7$	599C37	$13 \cdot 23$
137B7	$2 \cdot 17$	307B1	1	457C20	$2 \cdot 19$	601B29	$2 \cdot 5^2$
139A1	1	307C1	1	461D26	$5 \cdot 23$	607D31	101
139C7	23	307D1	1	463B22	$7 \cdot 11$	613C27	$3 \cdot 17$
149B9	37	307E2	3	467C26	233	617B28	$2 \cdot 7 \cdot 11$
151B3	1	307F9	17	479B32	239	619B30	103
151C6	5^2	311B22	$5 \cdot 31$	487A2	1	631B32	$3 \cdot 5 \cdot 7$

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