1.3 Modular Forms of Any Level

We next define spaces of modular forms of arbitrary level. For example, when $k = 2$ these are closely related to elliptic curves and abelian varieties.

A *congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$ is any subgroup that contains

$$\Gamma(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$$

for some N. The smallest such N is the *level* of Γ. For example,

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

are congruence subgroups of level N.

Definition 1.3.1 (Weakly Modular Function)**.** A *weakly modular function* of weight k for a congruence subgroup Γ is a meromorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that $f|[\gamma]_k = f$ for all $\gamma \in \Gamma$.

A central object in the theory of modular forms (and modular symbols) is the set of all *cusps*

$$\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}.$$

The set of cusps for a congruence subgroup Γ is the set $C(\Gamma)$ of orbits of $\mathbb{P}^1(\mathbb{Q})$ under the action of Γ. (We will often identify elements of $C(\Gamma)$ with a representative element from the orbit.) For example, if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ there is exactly one orbit.

Lemma 1.3.2. *For any cusps $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(\alpha) = \beta$.*

Proof. This is Exercise 7.7. □

Proposition 1.3.3. *For any congruence subgroup Γ, the set $C(\Gamma)$ of cusps is finite.*

In order to define modular forms for general congruence subgroups we need to make sense of what it means for a function to be holomorphic on the *extended upper halfplane*

$$\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}).$$

In Section 1.2 we considered a weakly modular function f on $\mathrm{SL}_2(\mathbb{Z})$ to be holomorphic at ∞ if its q-expansion is of the form $\sum_{n=0}^{\infty} a_n q^n$.

In order to make sense of holomorphicity of a weakly modular function f for Γ at any $\alpha \in \mathbb{Q}$, we first prove a lemma.

Lemma 1.3.4. *If $f : \mathfrak{h} \to \mathbb{C}$ is a weakly modular function of weight k for a congruence subgroup Γ, and $\delta \in \mathrm{SL}_2(\mathbb{Z})$, then $f|[\delta]_k$ is a weakly modular function for $\delta^{-1}\Gamma\delta$.*

Proof. If $s = \delta^{-1}\gamma\delta \in \delta^{-1}\Gamma\delta$, then

$$(f|[\delta]_k)|[s]_k = f|[\delta s]_k = f|[\delta\delta^{-1}\gamma\delta]_k = f|[\gamma\delta]_k = f|[\delta]_k.$$

□

Fix a weight k weakly modular function f for some congruence subgroup Γ, and suppose $\alpha \in \mathbb{Q}$. In Section 1.2 we constructed the q-expansion of f by using that $f(z) = f(z+1)$, which held since $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Unfortunately, there are congruence subgroups Γ such that $T \not\in \Gamma$. Moreover, even if we are interested only in modular forms for $\Gamma_1(N)$, where we have $T \in \Gamma_1(N)$ for all N, we have to consider q-expansions at infinity for modular forms on groups $\delta^{-1}\Gamma_1(N)\delta$, and these need not contain T. Fortunately, $T^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N)$, so a congruence subgroup of level N contains T^N. Thus for our f we have $f(z+h) = f(z)$ for some positive integer h, and again when f is meromorphic at infinity we obtain a Fourier expansion

$$f(z) = \sum_{n=m}^{\infty} a_n q^{n/N}, \tag{1.3.1}$$

but instead it is in powers of the function $q^{1/N} = e^{2\pi i z/N}$. We say that f is holomorphic at ∞ if in (1.3.1) we have $m \geq 0$.

What about the other cusps $\alpha \in \mathbb{P}^1(\mathbb{Q})$? By Lemma 1.3.2 there is a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(\infty) = \alpha$. We declare f to be *holomorphic at the cusp* α if the weakly modular function $f|[\gamma]_k$ is holomorphic at ∞.

Definition 1.3.5 (Modular Form)**.** A *modular form* of integer weight k for a congruence subgroup Γ is a weakly modular function $f : \mathfrak{h}^* \to \mathbb{C}$ that is holomorphic on $\mathfrak{h}^*$.

Proposition 1.3.6. *A weakly modular function f of weight k for Γ is holomorphic at every element of $\mathbb{P}^1(\mathbb{Q})$ if is holomorphic at a set of representative elements for $C(\Gamma)$.*

Proof. Let $c_1, \ldots, c_n \in \mathbb{P}^1(\mathbb{Q})$ be representatives for the set of cusps for Γ. If $\alpha \in \mathbb{P}^1(\mathbb{Q})$ then there is $\gamma \in \Gamma$ such that $\alpha = \gamma(c_i)$ for some i. By hypothesis f is holomorphic at c_i, so if $\delta \in \mathrm{SL}_2(\mathbb{Z})$ is such that $\delta(\infty) = c_i$, then $f|[\delta]_k$ is holomorphic at ∞. Since f is a weakly modular function,

$$f|[\delta]_k = (f|[\gamma]_k)|[\delta]_k = f|[\gamma\delta]_k. \tag{1.3.2}$$

But $\gamma(\delta(\infty)) = \gamma(c_i) = \alpha$, so (1.3.2) implies that f is holomorphic at α. □

1.3.1 Computing Widths of Cusps

Let Γ be a congruence subgroup of level N. Suppose $\alpha \in C(\Gamma)$ is a cusp, and choose $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(\infty) = \alpha$. The minimal h such that $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \in \gamma^{-1}\Gamma\gamma$ is called the *width* of the cusp α for the group Γ. In this section we discuss how to compute h.

Algorithm 1.3.7 (Width of Cusp)**.**
Given a congruence subgroup Γ of level N and a cusp α for Γ, this algorithm computes the width h of α. We assume that Γ is given by congruence conditions, e.g., $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$.

- [Find γ] Use the extended Euclidean algorithm to find $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(\infty) = \alpha$, as follows. If $\alpha = \infty$ set $\gamma = 1$; otherwise, write $\alpha = a/b$, find c, d such that $ad - bc = 1$, and set $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.
- [compute Conjugate Matrix] Compute the following matrix in $M_2(\mathbb{Z}[x])$:

$$\delta(x) = \gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \gamma^{-1}.$$

  Note that the entries of $\delta(x)$ are constant or linear in x.
- [Solve] The congruence conditions that define Γ give rise to four linear congruence conditions on x. Use techniques from elementary number theory (or enumeration) to find the smallest simultaneous positive solution h to these four equations.

Example 1.3.8. 1. Suppose $\alpha = 0$ and $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$. Then $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ has the property that $\gamma(\infty) = \alpha$. Next, the congruence condition is

$$\delta(x) = \gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}.$$

Thus the smallest positive solution is $h = N$, so the width of 0 is N.

2. Suppose $N = pq$ where p, q are distinct primes, and let $\alpha = 1/p$. Then $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$ sends ∞ to α. The congruence condition for $\Gamma_0(pq)$ is

$$\delta(x) = \gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1 - px & x \\ -p^2x & px + 1 \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{pq}.$$

Since $p^2x \equiv 0 \pmod{pq}$, we see that $x = q$ is the smallest solution. Thus $1/p$ has width q, and symmetrically $1/q$ has width p.

Remark 1.3.9. For $\Gamma_0(N)$, once we enforce that the bottom left entry is 0 (mod N), and use that the determinant is 1, the coprimality from the other two congruences is automatic. So there is one congruence to solve in the $\Gamma_0(N)$ case. There are 2 congruences in the $\Gamma_1(N)$ case.

1.4 Eisenstein Series and Delta

For an even integer $k \geq 4$, define the (non-normalized) *weight k Eisenstein series* to be

$$G_k(z) = \sum_{m,n\in\mathbb{Z}}^{*} \frac{1}{(mz+n)^k},$$

where the sum is over all $m, n \in \mathbb{Z}$ such that $mz + n \neq 0$.

Proposition 1.4.1. *The function $G_k(z)$ is a modular form of weight k.*

See [Ser73, § VII.2.3], where he proves that $G_k(z)$ defines a holomorphic function on $\mathfrak{h} \cup \{\infty\}$. To see that G_k is modular, note that

$$G_k(z+1) = \sum^{*} \frac{1}{(m(z+1)+n)^k} = \sum^{*} \frac{1}{(mz+(n+m))^k} = \sum^{*} \frac{1}{(mz+n)^k},$$

and

$$G_k(-1/z) = \sum^{*} \frac{1}{(-m/z+n)^k} = \sum^{*} \frac{z^k}{(-m+nz)^k} = z^k \sum^{*} \frac{1}{(mz+n)^k} = z^k G_k(z).$$

Proposition 1.4.2. *$G_k(\infty) = 2\zeta(k)$, where ζ is the Riemann zeta function.*

Proof. Taking the limit as $z \to i\infty$ in the definition of $G_k(z)$, we obtain $\sum^{*}_{n\in\mathbb{Z}} \frac{1}{n^k}$, since the terms involving z all go to 0 as $z \mapsto i\infty$. This sum is twice $\zeta(k) = \sum_{n\geq 1} \frac{1}{n^k}$. $\square$

For example, one can show that

$$G_4(\infty) = 2\zeta(4) = \frac{1}{3^2 \cdot 5}\pi^4$$

and

$$G_6(\infty) = 2\zeta(6) = \frac{2}{3^3 \cdot 5 \cdot 7}\pi^6.$$

Suppose $E = \mathbb{C}/\Lambda$ is an elliptic curve over $\mathbb{C}$, viewed as a quotient of $\mathbb{C}$ by a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, with $\omega_1/\omega_2 \in \mathfrak{h}$. Then

$$\wp_\Lambda(u) = \frac{1}{u^2} + \sum_{k=4,\ \text{even}}^{\infty} (k-1)G_k(\omega_1/\omega_2)u^{k-2},$$

and

$$(\wp')^2 = 4\wp^3 - 60G_4(\omega_1/\omega_2)\wp - 140G_6(\omega_1/\omega_2).$$

The discriminant of the cubic $4x^3 - 60G_4(\omega_1/\omega_2)x - 140G_6(\omega_1/\omega_2)$ is $16\Delta(\omega_1/\omega_2)$, where

$$\Delta = (60G_4)^3 - 27(140G_6)^2$$

is a cusp form of weight 12. Since E is an elliptic curve, $\Delta(\omega_1/\omega_2) \neq 0$.

Proposition 1.4.3. *For every even integer $k \geq 4$, we have*

$$G_k(z) = 2\zeta(k) + 2 \cdot \frac{(2\pi i)^k}{(k-1)!} \cdot \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_d(n)$ is the sum of the dth powers of the divisors of n.

For the proof, see [Ser73, §VII.4], which uses clever manipulations of various series, starting with the identity

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} + \frac{1}{z-m} \right).$$

From a computational point of view, the q-expansion for G_k from Proposition 1.4.3 is unsatisfactory, because it involves transcendental numbers. To understand more clearly what is going on, we introduce the *Bernoulli numbers* B_n for $n \geq 0$ *defined* by the following equality of formal power series:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{1.4.1}$$

Expanding the power series on the left we have

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \cdots$$

As this expansion suggests, the Bernoulli numbers B_n with $n > 1$ odd are 0 (see Exercise 7.6). Expanding the series further, we obtain the following table:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},$$

$$B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad B_{18} = \frac{43867}{798},$$

$$B_{20} = -\frac{174611}{330}, \quad B_{22} = \frac{854513}{138}, \quad B_{24} = -\frac{236364091}{2730}, \quad B_{26} = \frac{8553103}{6}.$$

For us the significance of the Bernoulli numbers is their connection with values of ζ at positive even integers.

Proposition 1.4.4. *If $k \geq 2$ is an* **even** *integer, then*

$$\zeta(k) = -\frac{(2\pi i)^k}{2 \cdot k!} \cdot B_k.$$

The proof involves manipulating a power series expansion for $z \cot(z)$ (see [Ser73, §VII.4]).

Definition 1.4.5 (Normalized Eisenstein Series)**.** The *normalized Eisenstein series* of even weight $k \geq 4$ is

$$E_k = \frac{(k-1)!}{2 \cdot (2\pi i)^k} \cdot G_k$$