Explicit Approaches to Elliptic Curves and Modular Forms

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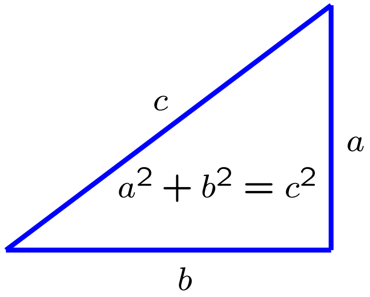
Outline of Course and this Lecture

1. Pythagoras and Fermat
2. Mordell-Weil Groups and the BSD Conjecture
3. Modularity of Elliptic Curves
4. Computing Modular Forms

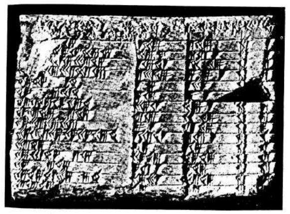
The Pythagorean Theorem



Pythagoras
Approx 569–475BC

Pythagorean Triples

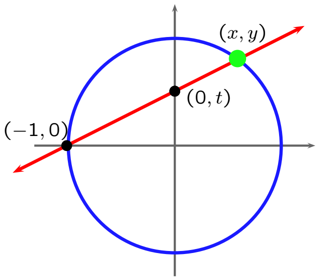
(3, 4, 5)
(5, 12, 13)
(7, 24, 25)
(9, 40, 41)
(11, 60, 61)
(13, 84, 85)
(15, 8, 17)
(21, 20, 29)
(33, 56, 65)
(35, 12, 37)
(39, 80, 89)
(45, 28, 53)
(55, 48, 73)
(63, 16, 65)
(65, 72, 97)
(77, 36, 85)
⋮



Triples of integers a, b, c such that

$$a^2 + b^2 = c^2$$

Enumerating Pythagorean Triples



$$\text{Slope} = t = \frac{y}{x+1}$$

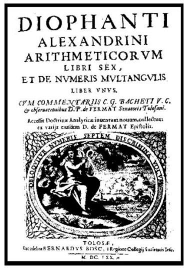
$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

If $t = \frac{r}{s}$, then $a = s^2 - r^2, \quad b = 2rs, \quad c = s^2 + r^2$
is a Pythagorean triple, and all primitive unordered triples arise in this way.

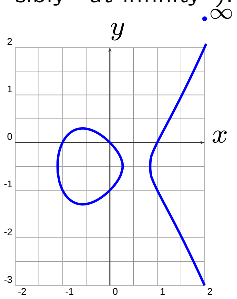
Fermat's "Last Theorem"

No analogue of "Pythagorean triples" with exponent 3 or higher.


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.

Wiles's Proof of FLT Uses Elliptic Curves

An **elliptic curve** is a nonsingular plane cubic curve with a rational point (possibly "at infinity").



$y^2 + y = x^3 - x$

EXAMPLES

$$y^2 + y = x^3 - x$$

$$x^3 + y^3 = 1 \text{ (Fermat cubic)}$$

$$y^2 = x^3 + ax + b$$

$$\cancel{3x^3 + 4y^3 + 5 = 0}$$

The Frey Elliptic Curve

Suppose Fermat's conjecture is **FALSE**. Then there is a prime $\ell \geq 5$ and coprime positive integers a, b, c with $a^\ell + b^\ell = c^\ell$.

Consider the corresponding Frey elliptic curve:

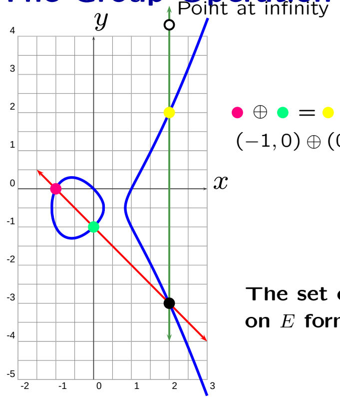
$$y^2 = x(x - a^\ell)(x + b^\ell).$$

Ribet's Theorem: This elliptic curve is not *modular*.

Wiles's Theorem: This elliptic curve is *modular*.

Conclusion: Fermat's conjecture is true.

The Group Operation



$y^2 + y = x^3 - x$

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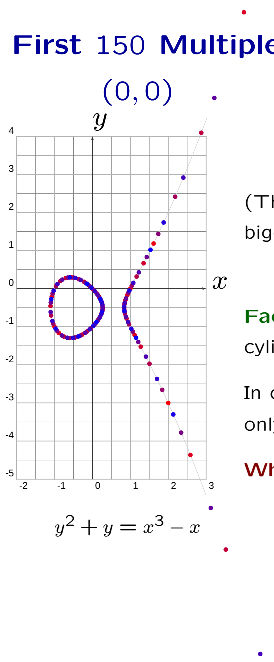
$(-1, 0) \oplus (0, -1) = (2, 2)$

The set of points on E forms an abelian group.

The First 150 Multiples of $(0, 0)$



$y^2 + y = x^3 - x$

(The bluer the point, the bigger the multiple.)

Fact: The group $E(\mathbf{Q})$ is infinite cylic, generated by $(0, 0)$.

In contrast, $y^2 + y = x^3 - x^2$ has only 5 rational points!

What is going on here?

Mordell's Theorem

Theorem (Mordell). The group $E(\mathbf{Q})$ of rational points on an elliptic curve is a **finitely generated abelian group**, so

$$E(\mathbf{Q}) \cong \mathbf{Z}^r \oplus T,$$

with $T = E(\mathbf{Q})_{\text{tor}}$ finite.

Mazur classified the possibilities for T. It is conjectured that r can be arbitrary, but the biggest r ever found is (probably) 24.

The Simplest Solution Can Be Huge

Simplest solution to $y^2 = x^3 + 7823$:

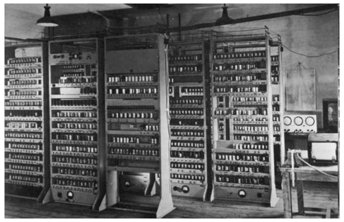
$$x = \frac{2263582143321421502100209233517777}{143560497706190989485475151904721}$$

$$y = \frac{186398152584623305624837551485596770028144776655756}{1720094998106353355821008525938727950159777043481}$$

(Found by Michael Stoll in 2002.)

The Central Question

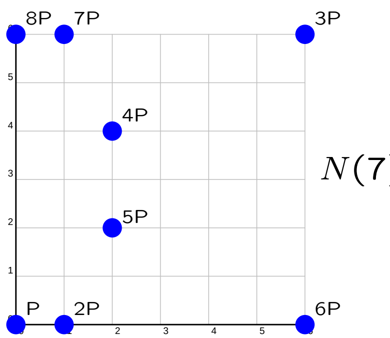
Given an elliptic curve E,
what is the rank of $E(\mathbb{Q})$?



Idea!: Consider the Group Modulo p

$N(p) = \#$ of solutions (mod p)

$y^2 + y = x^3 - x$ (mod 7)

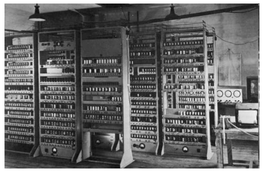


$$N(7) = 9$$

Counting Points

Cambridge EDSAC: The first
point counting supercomputer...

Birch and Swinnerton-Dyer



Hecke Eigenvalues

Let

$$a_p = p + 1 - N(p).$$

Hasse

Hasse proved that

$$|a_p| \leq 2\sqrt{p}.$$

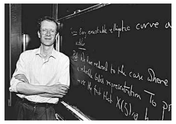
For $y^2 + y = x^3 - x$:

$a_2 = -2,\quad a_3 = -3,\quad a_5 = -2,\quad a_7 = -1,\quad a_{11} = -5,\quad a_{13} = -2,$

$a_{17} = 0,\quad a_{19} = 0,\quad a_{23} = 2,\quad a_{29} = 6,\quad \ldots$

Birch and Swinnerton-Dyer

The L-Function



Theorem (Wiles et al., Hecke) The following function extends to a holomorphic function on the whole complex plane:

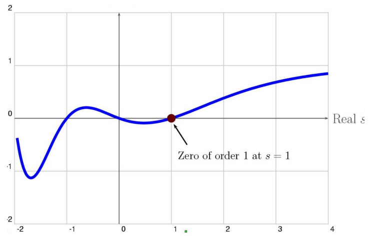
$$L^*(E,s) = \prod_{p\Delta} \left(\frac{1}{1 - a_p \cdot p^{-s} + p \cdot p^{-2s}} \right).$$

Here $a_p = p + 1 - \#E(\mathbf{F}_p)$ for all $p\Delta_E$. Note that formally,

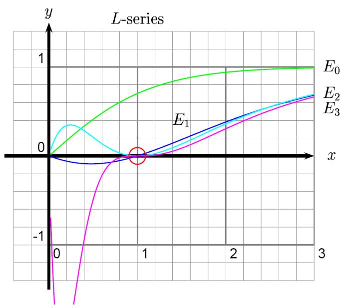
$$L^*(E,1) = \prod_{p\Delta} \left(\frac{1}{1 - a_p \cdot p^{-1} + p \cdot p^{-2}} \right) = \prod_{p\Delta} \left(\frac{p}{p - a_p + 1} \right) = \prod_{p\Delta} \frac{p}{N_p}$$

Standard extension to $L(E,s)$ at bad primes.

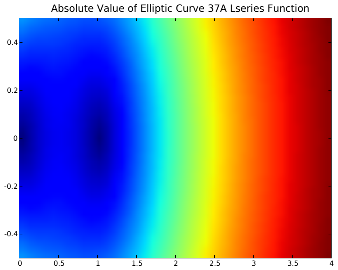
Real Graph of the L-Series of $y^2 + y = x^3 - x$

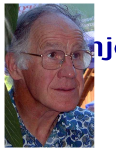


More Graphs of Elliptic Curve L-functions



Absolute Value of L-series on Complex Plane for $y^2 + y = x^3 - x$





njectures Proliferated

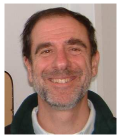
"The subject of this lecture is rather a special one. I want to describe some computations undertaken by myself and Swinnerton-Dyer on EDSAC, by which we have calculated the zeta-functions of certain elliptic curves. As a result of these computations we have found an analogue for an elliptic curve of the Tamagawa number of an algebraic group; and conjectures have proliferated. [...] though the associated theory is both abstract and technically complicated, the objects about which I intend to talk are usually simply defined and often machine computable; **experimentally we have detected certain relations between different invariants**, but we have been unable to approach proofs of these relations, which must lie very deep." – Birch 1965

The Birch and Swinnerton-Dyer Conjecture

Conjecture: Let E be any elliptic curve over $\mathbf{Q}$. The order of vanishing of $L(E, s)$ as $s = 1$ equals the rank of $E(\mathbf{Q})$.

The Kolyvagin and Gross-Zagier Theorem

Theorem: If the ordering of vanishing $\mathrm{ord}_{s=1}L(E, s)$ is ≤ 1, then the conjecture is true for E.



Elliptic Curves are "Modular"

An elliptic curve is **modular** if the numbers a_p are coefficients of a "modular form". Equivalently, if $L(E, s)$ extends to a complex analytic function on $\mathbf{C}$ (with functional equation).

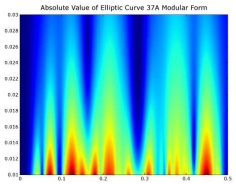
Theorem (Wiles et al.): *Every elliptic curve over the rational numbers is modular.*

Wiles at the Institute for Advanced Study

Modular Forms

The definition of modular forms as holomorphic functions satisfying a certain equation is very abstract.

For today, I will skip the abstract definition, and instead give you an explicit "engineer's recipe" for producing modular forms. In the meantime, here's a picture:



Computing Modular Forms: Motivation

Motivation: Data about modular forms is **extremely** useful to many research mathematicians (e.g., number theorists, cryptographers). This data is like the astronomer's telescope images.

One of my longterm research goals is to compute modular forms on a **huge** scale, and make the resulting database widely available. I have done this on a smaller scale during the last 5 years — see `http://modular.ucsd.edu/Tables/`

What to Compute: Newforms

For each positive integer N there is a finite list of **newforms** of level N. E.g., for $N = 37$ the newforms are

$$f_1 = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + \cdots$$
$$f_2 = q + q^3 - 2q^4 - q^7 + \cdots,$$

where $q = e^{2\pi i z}$.

The newforms of level N determine all the modular forms of level N (like a basis in linear algebra). The coefficients are algebraic integers. *Goal: compute these newforms.*

Bad idea – write down many elliptic curves and compute the numbers a_p by counting points over finite fields. No good – this misses most of the interesting newforms, and gets newforms of all kinds of random levels, but you don't know if you get everything of a given level.

An Engineer's Recipe for Newforms

Fix our positive integer N. For simplicity assume that N is prime.

1. Form the $N+1$ dimensional $\mathbf{Q}$-vector space V with basis the symbols $[0], \ldots, [N-1], [\infty]$.

2. Let R be the suspace of V spanned by the following vectors, for $x = 0, \ldots, N-1, \infty$:
$$[x] - [N-x]$$
$$[x] + [x.S]$$
$$[x] + [x.T] + [x.T^2]$$
$S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, and $x.\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (ax+c)/(bx+d)$.

3. Compute the quotient vector space $M = V/R$. This involves "intelligent" **sparse Gauss elimination** on a matrix with $N+1$ columns.

Implementation

- I implemented code for computing modular forms that's included with **MAGMA** (non-free, closed source): `http://magma.maths.usyd.edu.au/magma/`.

- I want something better, so I'm implementing modular symbols algorithms as part of **SAGE**: `http://modular.ucsd.edu/sage/`.

- I'm finishing a **book** on these algorithms that will be published by the American Mathematical Society.

4. Compute the matrix T_2 on M given by
$$[x] \mapsto [x.\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)] + [x.\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)] + [x.\left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)] + [x.\left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right)].$$
This matrix is unfortunately not sparse. Similar recipe for matrices T_n for any n.

5. Compute the **characteristic polynomial** f of T_2.

6. **Factor** $f = \prod g_i^{e_i}$. Assume all $e_i = 1$ (if not, use a random linear combination of the T_n.)

7. Compute the **kernels** $K_i = \ker(g_i(T_2))$. The **eigenvalues** of T_3, T_5, etc., acting on an **eigenvector** in K_i give the coefficients a_p of the newforms of level N.

The Modular Forms Database Project

- Create a database of all newforms of level N for each $N < 100000$. This will require many gigabytes to store. (50GB?)

- So far this has only been done for $N < 7000$ (and is incomplete), so 100000 is a **major challenge**.

- Involves sparse linear algebra over $\mathbf{Q}$ on spaces of dimension up to 200000 and dense linear algebra on spaces of dimension up to 25000.

- Easy to parallelize – run one process for each N.

- Will be very useful to number theorists and cryptographers.

- John Cremona has done something similar but only for the newforms corresponding to elliptic curves (he's at around 120000 right now), so this should be do-able.