Math 129: Algebraic Number Theory
Homework Assignment 9

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Due: Thursday, April 22, 2004

There are SIX problems.

1. In this problem you will compute an example of weak approximation, like I did in the example in Section 6 of lecture 18. Let $K = \mathbf{Q}$, let $|\cdot|_7$ be the 7-adic absolute value, let $|\cdot|_{11}$ be the 11-adic absolute value, and let $|\cdot|_\infty$ be the usual archimedean absolute value. Find an element $b \in \mathbf{Q}$ such that $|b - a_i|_i < \frac{1}{10}$, where $a_7 = 1$, $a_{11} = 2$, and $a_\infty = -2004$.

2. Prove that -9 has a cube root in $\mathbf{Q}_{10}$ using the following strategy (this is a special case of Hensel's Lemma, which you can read about in an appendix to Cassel's article).

   (a) Show that there is an element $\alpha \in \mathbf{Z}$ such that $\alpha^3 \equiv 9 \pmod{10^3}$.

   (b) Suppose $n \geq 3$. Use induction to show that if $\alpha_1 \in \mathbf{Z}$ and $\alpha^3 \equiv 9 \pmod{10^n}$, then there exists $\alpha_2 \in \mathbf{Z}$ such that $\alpha_2^3 \equiv 9 \pmod{10^{n+1}}$. (Hint: Show that there is an integer b such that $(\alpha_1 + b \cdot 10^n)^3 \equiv 9 \pmod{10^{n+1}}$.)

   (c) Conclude that 9 has a cube root in $\mathbf{Q}_{10}$.

3. Prove that every finite extension of $\mathbf{Q}_p$ "comes from" an extension of $\mathbf{Q}$, in the following sense. Given an irreducible polynomial $f \in \mathbf{Q}_p[x]$ there exists an irreducible polynomial $g \in \mathbf{Q}[x]$ such that the fields

$$\mathbf{Q}_p[x]/(f)$$

   and $\mathbf{Q}_p[x]/(g)$ are isomorphic. [Hint: Choose each coefficient of g to be sufficiently close to the corresponding coefficient of f, then use Hensel's lemma to show that g has a root in $\mathbf{Q}_p[x]/(f)$.]

4. Find the 3-adic expansion to precision 4 of each root of the following polynomial over $\mathbf{Q}_3$:

$$f = x^3 - 3x^2 + 2x + 3 \in \mathbf{Q}_3[x].$$

Your solution should conclude with three expressions of the form

$$a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3 + O(3^4).$$

5. (a) Find the normalized Haar measure of the following subset of $\mathbf{Q}_7^+$:

$$U = B\left(28, \frac{1}{50}\right) = \left\{x \in \mathbf{Q}_7 : |x - 28| < \frac{1}{50}\right\}.$$

(b) Find the normalized Haar measure of the subset $\mathbf{Z}_7^*$ of $\mathbf{Q}_7^*$.

6. Suppose that K is a finite extension of $\mathbf{Q}_p$ and L is a finite extension of $\mathbf{Q}_q$, with $p \neq q$ and assume that K and L have the same degree. Prove that there is a polynomial $g \in \mathbf{Q}[x]$ such that $\mathbf{Q}_p[x]/(g) \cong K$ and $\mathbf{Q}_q[x]/(g) \cong L$. [Hint: Combine your solution to 3 with the weak approximation theorem.]