Math 129: Algebraic Number Theory
Homework Assignment 6

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Due: Thursday, March 25, 2004

1. Let $p \in \mathbf{Z}$ and let K be a number field. Show that $\mathrm{Norm}_{K/\mathbf{Q}}(p\mathcal{O}_K) = p^{[K:\mathbf{Q}]}$.
2. A totally real number field is a number field in which all embeddings into $\mathbf{C}$ have image in $\mathbf{R}$. Prove there are totally real number fields of degree p, for every prime p. [Hint: Let ζ_n denote a primitive nth root of unity. For $n \geq 3$, show that $\mathbf{Q}(\zeta_n + 1/\zeta_n)$ is totally real of degree $\varphi(n)/2$. Now prove that $\varphi(n)/2$ can be made divisible by any prime.]
3. Give an example of a number field $K/\mathbf{Q}$ and a prime p such that the e_i in the factorization of $p\mathcal{O}_K$ are not all the same.
4. Let K be a number field. Give the "simplest" proof you can think of that there are only finitely many primes that ramify (i.e., have some $e_i > 1$) in K. [The meaning of "simplest" is a matter of taste.]
5. Give examples to show that for $K/\mathbf{Q}$ a Galois extension, the quantity e can be arbirarily large and f can be arbitrarily large.
6. Suppose $K/\mathbf{Q}$ is Galois and p is a prime such that $p\mathcal{O}_K$ is also prime (i.e., p is inert in K). Show that $\mathrm{Gal}(K/\mathbf{Q})$ is a cyclic group.
7. (Problem 7, page 116, from Marcus *Number Fields*) For each of the following, find a prime p and quadratic extensions K and L of $\mathbf{Q}$ that illustrates the assertion:
   (a) The prime p can be totally ramified in K and L without being totally ramified in KL.
   (b) The fields K and L can each contain unique primes lying over p while KL does not.
   (c) The prime p can be inert in K and L without being inert in KL.
   (d) The residue field extensions of $\mathbf{F}_p$ can be trivial for K and L without being trivial for KL.