Math 129: Algebraic Number Theory
Homework Assignment 5

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Due: Thursday, March 18, 2004

1. Prove that any finite subgroup of the multiplicative group of a field is cyclic.

2. For a given number field K, which seems more difficult for MAGMA to compute, the class groups or explicit generators for the group of units? It is very difficult (but not impossible) to not get full credit on this problem. Play around with some examples, see what seems more difficult, and *justify* your response with examples. (This problem might be annoying to do using the MAGMA web page, since it kills your MAGMA job after 30 seconds. Feel free to request a binary of MAGMA from me, or an account on MECCAH (Mathematics Extreme Computation Cluster at Harvard).)

3. (a) Prove that there is no number field K such that $U_K \cong \mathbf{Z}/10\mathbf{Z}$.
   (b) Is there a number field K such that $U_K \cong \mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$?

4. Prove that the rank of U_K is unbounded as K varies over all number fields.

5. Let $K = \mathbf{Q}(\zeta_5)$.
   (a) Show that $r = 0$ and $s = 2$.
   (b) Find explicitly generators for the group of units of U_K (you can use MAGMA for this).
   (c) Draw an illustration of the log map $\varphi : U_K \to \mathbf{R}^2$, including the hyperplane $x_1 + x_2 = 0$ and the lattice in the hyperplane spanned by the image of U_K.

6. Find the group of units of $\mathbf{Q}(\zeta_n)$ as an abstract group as a function of n. (I.e., find the number of cyclic factors and the size of the torsion subgroup. You do not have to find explicit generators!)

7. Let $K = \mathbf{Q}(a)$, where a is a root $x^3 - 3x + 1$.
   (a) Show that $r = 3$.
   (b) Find explicitly the log embedding of U_K into a 2-dimensional hyperplane in $\mathbf{R}^3$, and draw a picture.

8. Prove that if K is a quadratic field and the torsion subgroup of U_K has order bigger than 2, then $K = \mathbf{Q}(\sqrt{-3})$ or $K = \mathbf{Q}(\sqrt{-1})$.

9. A *Salem number* is a real algebraic integer, greater than 1, with the property that all of its conjugates lie on or within the unit circle, and at least one conjugate lies on the unit circle. By any method (including "google"), give two examples of Salem numbers.