Harvard Math 129: Algebraic Number Theory
Homework Assignment 6

William Stein

Due: TUESDAY, March 22, 2005

Note the unusual due date, which is to avoid making the homework due right before spring break.

The problems have equal point value, and multi-part problems are of the same value. In any problem where you use a computer, include in your solution the exact commands you type and their output. You may use any software, including (but not limited to) MAGMA and PARI.

For a number field K, let U_K denote the group $\mathcal{O}_K^*$ of units of its ring of integers. In each of the following problems, answer the question and give a proof that your answer is correct.

1. Let K vary through all number fields. What torsion subgroups $(U_K)_{\mathrm{tor}}$ actually occur?

2. If $U_K \approx \mathbb{Z}^n \times (U_K)_{\mathrm{tor}}$, we say that U_K has rank n. Let K vary through all number fields. What ranks actually occur?

3. Let K vary through all number fields such that the group U_K of units of K is a finite group. What finite groups U_K actually occur?

4. Let $K = \mathbb{Q}(\zeta_5)$.
   (a) Show that $r = 0$ and $s = 2$.
   (b) Find explicit generators for the group of units of U_K.
   (c) Draw an illustration of the log map $\varphi : U_K \to \mathbb{R}^2$, including the hyperplane $x_1 + x_2 = 0$ and the lattice in the hyperplane spanned by the image of U_K.