Harvard Math 129: Algebraic Number Theory
Homework Assignment 2

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Due: Thursday, February 24, 2005

The problems have equal point value, and multi-part problems are of the same value. There are **7** *problems.*

1. Prove that the ring $\overline{\mathbb{Z}}$ is not noetherian, but it is integrally closed in its field of fraction, and every nonzero prime ideal is maximal. Thus $\overline{\mathbb{Z}}$ is not a Dedekind domain.

2. Let K be a field.
   (a) Prove that the polynomial ring $K[x]$ is a Dedekind domain.
   (b) Is $\mathbb{Z}[x]$ a Dedekind domain?

3. Let $\mathcal{O}_K$ be the ring of integers of a number field. Let F_K denote the abelian group of fractional ideals of $\mathcal{O}_K$.
   (a) Prove that F_K is torsion free.
   (b) Prove that F_K is not finitely generated.
   (c) Prove that F_K is countable.
   (d) Conclude that if K and L are number fields, then there exists an isomorphism of groups $F_K \approx F_L$.

4. From basic definitions, find the rings of integers of the fields $\mathbb{Q}(\sqrt{11})$ and $\mathbb{Q}(\sqrt{13})$.

5. Factor the ideal (10) as a product of primes in the ring of integers of $\mathbb{Q}(\sqrt{11})$. You're allowed to use a computer, as long as you show the commands you use.

6. Let $\mathcal{O}_K$ be the ring of integers of a number field K, and let $p \in \mathbb{Z}$ be a prime number. What is the cardinality of $\mathcal{O}_K/(p)$ in terms of p and $[K : \mathbb{Q}]$, where (p) is the ideal of $\mathcal{O}_K$ generated by p? (Prove that your formula is correct.)

7. Give an example of each of the following, with proof:

   (a) A non-principal ideal in a ring.

   (b) A module that is not finitely generated.

   (c) The ring of integers of a number field of degree 3.

   (d) An order in the ring of integers of a number field of degree 5.

   (e) The matrix on K of left multiplication by an element of K, where K is a degree 3 number field.

   (f) An integral domain that is not integrally closed in its field of fractions.

   (g) A Dedekind domain with finite cardinality.

   (h) A fractional ideal of the ring of integers of a number field that is not an integral ideal.