Math 129: Algebraic Number Theory
Homework Assignment 1

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Due: Thursday, February 17, 2005

The problems have equal point value, and multi-part problems are of the same value.

1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.
   (a) Find the Smith normal form of A.
   (b) Prove that the cokernel of the map $\mathbb{Z}^3 \to \mathbb{Z}^3$ given by multiplication by A is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}$.

2. Show that the minimal polynomial of an algebraic number $\alpha \in \overline{\mathbb{Q}}$ is unique. (You may assume that $\mathbb{Q}[x]$ is a prinicipal ideal domain, which is easy to prove using the division algorithm for polynomials.)

3. Which of the following rings have infinitely many prime ideals? (Prove that your answers are correct.)
   (a) The integers $\mathbb{Z}$.
   (b) The ring $\mathbb{Z}[x]$ of polynomials over $\mathbb{Z}$.
   (c) The quotient ring $\mathbb{C}[x]/(x^{2005} - 1)$.
   (d) The ring $(\mathbb{Z}/6\mathbb{Z})[x]$ of polynomials over the ring $\mathbb{Z}/6Z$.
   (e) The quotient ring $\mathbb{Z}/n\mathbb{Z}$, for a fixed positive integer n.
   (f) The rational numbers $\mathbb{Q}$.
   (g) The polynomial ring $\mathbb{Q}[x, y, z]$ in three variables.

4. Which of the following numbers are algebraic integers?
   (a) The number $(1 + \sqrt{5})/2$.
   (b) The number $(2 + \sqrt{5})/2$.
   (c) The value of the infinite sum $\sum_{n=1}^{\infty} 1/n^2$.
   (d) The number $\alpha/3$, where α is a root of $x^4 + 54x + 243$.

5. Prove that $\overline{\mathbb{Z}}$ is not noetherian.